# Percolation on the triangular lattice

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## 1 Introduction

Percolation theory is concerned with the connectivity properties of random graphs. Historically, its main goal was to give a mathematical model for the study of flows of liquid in random media. Since it is difficult to give a accurate generic definition of the subject we focus immediatly on the particular, but fundamental, example of model which will be discussed in this report.

### 1.1 Independent site percolation

Suppose that $G$ is a graph on which the following random experiment is made: given a number $p \in [0, 1]$, each vertex of the graph is randomly choosen to be either *open* or *closed* with probability $p$ and $1-p$ respectively, independently of the other vertices. This process is called *independent site percolation* (by opposition to *bond percolation* (see [4] for an introduction to the subject, for instance), where the same process is considered on the edges of the graph).

The subject of study of percolation theory is the existence, shape and size of open *paths* (paths formed by open vertices) and *clusters* (connected components consisting of open vertices) appearing in such a random graph. This theory is interested in questions such as:

- Does an infinite open cluster exist?
- What is the average size of an open cluster?
- What is the probability that given parts of the graph are joined by open paths?

Although elementary in their formulation these problems turn out to be really difficult to investigate and their answers, when they are known, often involve far less elementary notions. A good example of this fact is the Cardy's conjecture which relates such questions to the theory of complex functions, and which has been proved in a particular case by Smirnov. The aim of this report is to present the methods used to establish this surprising link.

### 1.2 The triangular lattice

Smirnov's theorem shows conformal invariance of the limits of probabilities of certain events on the independent site percolation equilateral on triangular lattice with parameter $p = \frac{1}{2}$.

Let us define this notions briefly.

We denote by $\mathcal{G}_\delta$ the *equilateral triangular lattice* embedded in the complex plane $\mathbb{C}$ with *mesh* (or scale factor) $\delta > 0$, spanned by the third roots of unity $\mathcal{C}_3 = \{1, j = e^{2\pi i/3}, j^2 = e^{4\pi i/3}\}$ (see Figure 1).

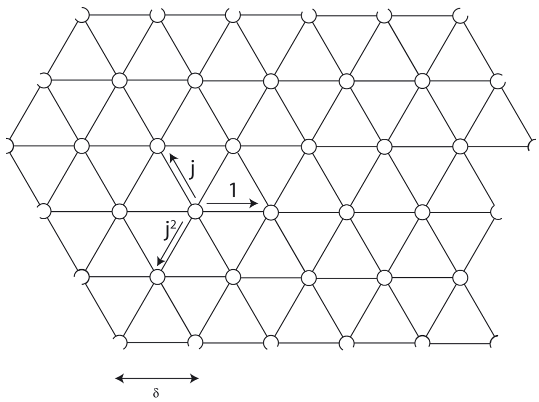


Figure 1: The equilateral triangular lattice

Now we consider the percolation process with parameter $\frac{1}{2}$ on $\mathcal{G}_\delta$. It is easy to see that this is equivalent to a random colouring of the faces of the regular hexagonal lattice: each vertex of $\mathcal{G}_\delta$ can be seen naturally as the center of a hexagonal face (the hexagonal lattice is called the *dual* of $\mathcal{G}_\delta$), and so we can colour a face in *white* (respectively *black*) if the corresponding vertex is open (respectively closed). Because of the symmetry of the problem (the parameter $p$ equals $\frac{1}{2}$) we will prefer this formulation in what follows.

Let $\mathcal{R}$ be a *conformal rectangle*, that is, a domain bounded by a Jordan curve, with (in counterclockwise order) four points, called *vertices*, $a, b, c, d$ on its boundary. We discretise $\mathcal{R}$ by taking the largest connected component of its intersection with the lattice $\mathcal{G}_\delta$ such that each vertex is the center of a face contained in $\mathcal{R}$ (we do not take the vertices on the boundary with the some part of the associated hexagon lying outside $\mathcal{R}$, see Figure 3) associating to each of the four points the nearest vertex in the discretization, which we denote by $a_\delta, b_\delta, c_\delta, d_\delta$. It is easy to see that the shape of the discretization approximates the rectangle well for a small mesh $\delta$.

In the discretised rectangle we consider the probability of the event that there is a white path *separating* $a_\delta$ and $b_\delta$ from $c_\delta$ and $d_\delta$, in the sense that

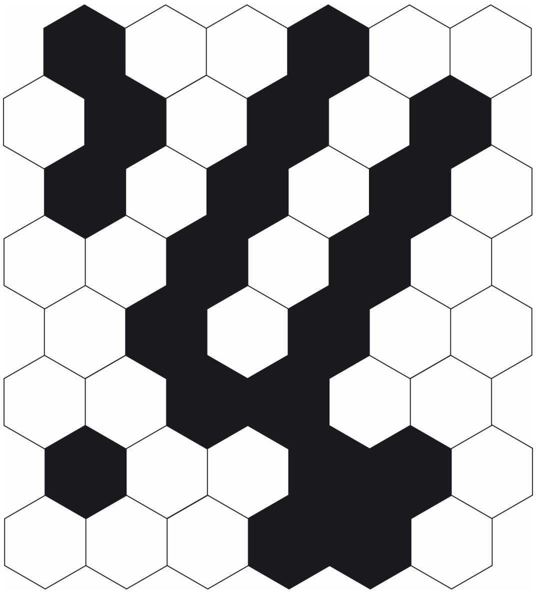

Figure 2: Percolation seen as coloring of hexagons

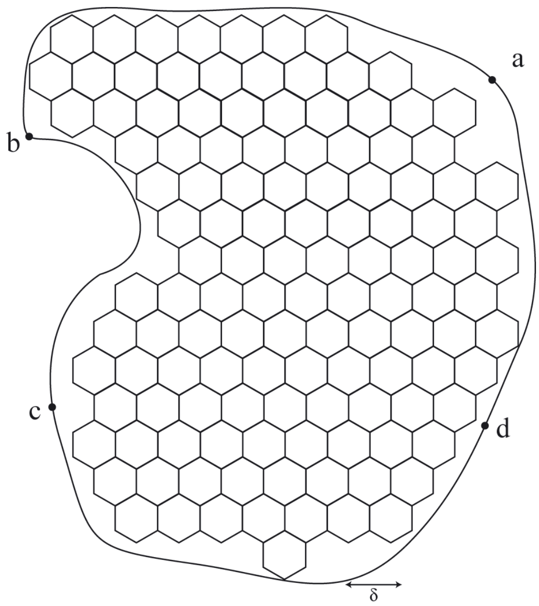


Figure 3: Discretization of a conformal rectangle

any path joining $a_\delta$ or $b_\delta$ to $c_\delta$ or $d_\delta$ inside the discretised rectangle must cross the white path. We call a such event a *crossing event*. The probability of this event depends on the shape of the rectangle, on the four points and on the mesh $\delta$ and we denote it by $C_\delta\left(\mathcal{R}, a, b, c, d\right)$.

Notice a fundamental property of the equilateral triangular lattice, called *self-duality*: if there is no white path separating $a_\delta$ and $b_\delta$ from $c_\delta$ and $d_\delta$, then there exists a black path separating $a_\delta$ and $d_\delta$ from $b_\delta$ and $c_\delta$. Both events cannot occur simultaneously, so they form a partition of the probability space.

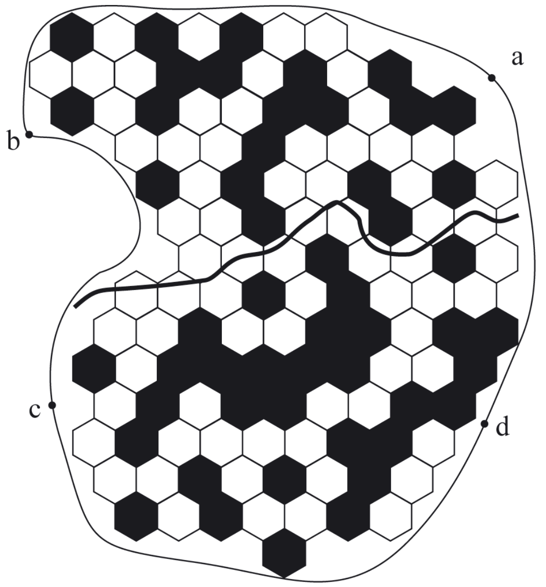


Figure 4: A white path separating $a$ and $b$ from $c$ and $d$

Smirnov's theorem (see [9]) characterizes the way that this probability depends on the rectangle $\mathcal{R}$. Recall that a *conformal mapping* is a bijective holomorphic function (its inverse is in fact automatically holomorphic) between two domains of $\mathbb{C}$.

We say that two conformal rectangles $\mathcal{R}$ and $\mathcal{R}'$ are *conformally equivalent* if there is a conformal mapping from the domain of $\mathcal{R}$ to the domain of $\mathcal{R}'$, that extends continuously to the boundary of $\mathcal{R}$ and maps the four points $a, b, c, d$ of the $\mathcal{R}$ to the four points $a', b', c', d'$ of $\mathcal{R}'$, in that order. This is an equivalence relation.

**Theorem 1** (Smirnov)**.** *The probability $C_\delta(\mathcal{R}, a, b, c, d)$ admits a limit as the mesh $\delta$ tends to $0$, which is conformally invariant: for every rectangle $(\mathcal{R}', a', b', c', d')$ which is conformally equivalent to $(\mathcal{R}, a, b, c, d)$, the limit is the same,*

$$\lim_{\delta \to 0} C_\delta(\mathcal{R}, a, b, c, d) = \lim_{\delta \to 0} C_\delta\left(\mathcal{R}', a', b', c', d'\right).$$

To prove this theorem, we establish a reformulation in terms of *conformal triangles*, which are also domains bounded by Jordan curves, with three vertices (three points on the boundary). Let us denote by $\mathcal{T}$ a conformal triangle and by $a_1, a_j, a_{j^2}$ the corresponding vertices (notice that for symmetry reasons the indices are choosen as the third roots of unity); the three vertices of $\mathcal{T}$ partition its boundary into three *sides* denoted by $A_1$, $A_j$ and $A_{j^2}$ which are the ones that do not contain (or, equivalently, are opposite to) $a_1$, $a_j$ and $a_{j^2}$ respectively. Again, we discretize $\mathcal{T}$ into a subgraph $\mathcal{T}_\delta$ and associate to the points $a_1$, $a_j$, $a_{j^2}$ the nearest vertices of $\mathcal{T}_\delta$ $\mathcal{T}_\delta$ which are denoted by $a_1^\delta$, $a_j^\delta$, $a_{j^2}^\delta$.

As before, we are interested in crossing events. For every point $z$ inside $\mathcal{T}$, let $Q_1^\delta(z)$ be the event that there exists a white path separating $a_1^\delta$ and $z$ from $a_j^\delta$ and $a_{j^2}^\delta$, and define the events $Q_j^\delta(z)$ and $Q_{j^2}^\delta(z)$ similarly, by rotating the indices (the index of the event corresponds to the point that is on the same side of the path as $z$). Let $H_1^\delta(z)$, $H_j^\delta(z)$, $H_{j^2}^\delta(z)$ be the respective probabilities of the events $Q_1^\delta(z), Q_j^\delta(z), Q_{j^2}^\delta(z)$.

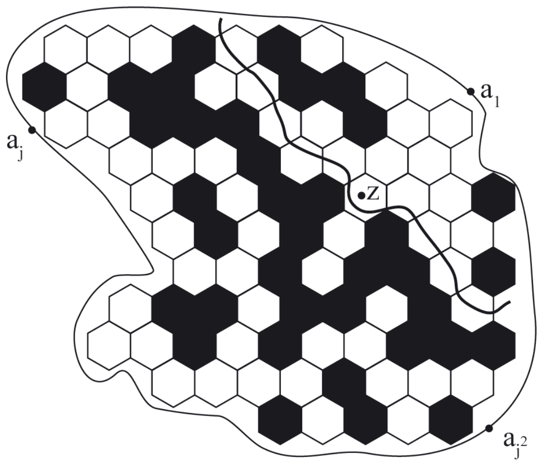


Figure 5: A white path separating $a_\delta$ and $z$ from $c_\delta$ and $d_\delta$

**Remark 1.** *The probability $C_\delta(\mathcal{R}, a, b, c, d)$ on the rectangle with four point $a, b, c, d$ can be seen as the value of $Q_1^\delta$, taking the same domain, letting $a_1 = a, a_j = c, a_{j^2} = d$ and and evaluating at $z = b$ (after having continuously extended the function $Q_1^\delta$ to the boundary of $\mathcal{T}$).*

Conformal equivalence for triangles is defined the same way as for rectangles, the difference being that existence and uniqueness of conformal mapping between triangles is ensured by the following theorem (see [8], on conformal mapping, for instance).

**Theorem 2.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be two conformal triangles. Then there exists a unique conformal mapping $\Phi_{\mathcal{T}}^{\mathcal{T}'}$ that maps the domain and the vertices of $\mathcal{T}$ to the ones of $\mathcal{T}'$.*

We are now able to give a reformulation of Smirnov's theorem's conformal invariance in terms of triangles.

**Theorem 3.** *The functions $H_1^\delta, H_j^\delta, H_{j^2}^\delta$ converge uniformly as $\delta$ tends to 0. The limits, call them $h_1, h_j, h_{j^2}$ respectively, are conformally invariant: let $\mathcal{T}'$ be another conformal triangle, with the corresponding functions $h_1', h_j', h_{j^2}'$. Then, if we denote by $\Phi_{\mathcal{T}}^{\mathcal{T}'} : \mathcal{T} \to \mathcal{T}'$ the conformal mapping between $\mathcal{T}$ and $\mathcal{T}'$, we have that*

$$h_1 = h_1' \circ \Phi_{\mathcal{T}}^{\mathcal{T}'}, \quad h_j = h_j' \circ \Phi_{\mathcal{T}}^{\mathcal{T}'}, \quad h_{j^2} = h_{j^2}' \circ \Phi_{\mathcal{T}}^{\mathcal{T}'}.$$

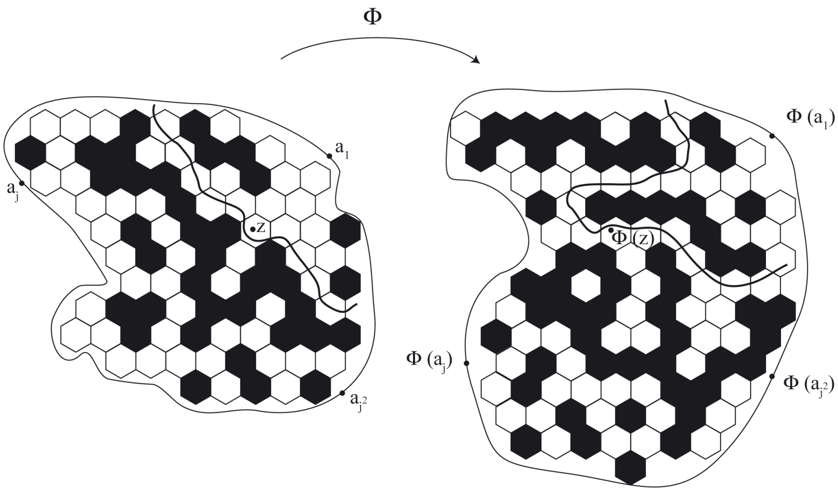


Figure 6: The limits of the probabilities are conformally invariant

It is easy to see that the original theorem is a particular case of this theorem, by the remark above and noticing that conformal mappings between rectangles are in particular between triangles.

In fact, this is not the latter formulation we will prove in this report, but a more geometrical and beautiful one, which moreover gives a simple estimation of the probability when the domain is an equilateral triangle. Recall that $j$ is the third root of unity equal to $e^{2\pi i/3}$ and that we denote by $\mathcal{C}_3 = \{1, j, j^2\}$ the group generated by $j$.

**Theorem 4.** *With the same notations as above, the limits of $h_1, h_j, h_{j^2}$ exist and if we define the function $f$ and $g$ as*

$$f = \sum_{\mu \in \mathcal{C}_3} h_\mu, \quad g = \sum_{\mu \in \mathcal{C}_3} \left( \frac{1}{3} + \frac{2}{3}\mu \right) h_\mu,$$

*then $f$ is identically equal to 1 and $g$ is the unique conformal mapping from the conformal triangle $\mathcal{T}$ to the equilateral triangle $\Delta$ whose vertices are $1, \frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}$ (in that order).*

Uniqueness of the conformal mapping gives us conformal invariance of $g$ (in the same sense as in the previous formulation) immediatly: if $\Phi_{\mathcal{T}}^{\mathcal{T}'}$ is a conformal mapping between $\mathcal{T}$ and $\mathcal{T}'$, with $g$ defined as above for $\mathcal{T}$ and $g'$ similarly for $\mathcal{T}'$, then as $g' \circ \Phi_{\mathcal{T}}^{\mathcal{T}'}$ is a conformal mapping from $\mathcal{T}$ to $\Delta$, it unique and therefore equal to $g$.

Taking the real part of $g$ yields $h_1$ (the factors of $h_j$ and $h_{j^2}$ are purely imaginary), so $h_1$ is conformally invariant, and we obtain in a similar way the same result for $h_j$ and $h_{j^2}$, implying Theorem 2.

Let us finally mention a direct consequence of this theorem, which is the simple form that arise naturally on the equilateral triangle $\Delta$.

**Corollary 1** (Carleson's reformulation)**.** *If we choose our conformal triangle $\mathcal{T}$ to be actually the equilateral triangle $\Delta$ defined above, we obtain that the limit $h_1$ is given by the following expression:*

$$h_1(z) = \mathrm{Re}(z).$$

*By symmetry, similar equations hold for $h_j(z)$ and $h_{j^2}(z)$.*

This follows from the fact that $g$ is equal, by uniqueness, in this case to the identity function. As discussed a few lines above $h_1(z)$ is equal to $\mathrm{Re}\,(g(z)) = \mathrm{Re}(z)$.

It is really surprising to see a linear combination of three probabilities defined on discrete events tending to a conformal mapping. This shows that the analytic functions theory lies behind the apparently disordered phenomenon of percolation on the triangular lattice and gives us a deeper understanding of this process.

From the point of view of complex analysis this theorem shows in a very constructive way the existence of a conformal mapping from any domain bounded by a Jordan curve to the equilateral triangle $\Delta$, which is given for instance by Riemann's mapping theorem under more general hypothesis (simple connectivity), but in general in a far less constructive way.

### 1.3 Proof of Theorem 4

The proof of Theorem 4 involves essentially three technical results which will be discussed in detail in this report. Let us suppose that we are on the conformal triangle $\mathcal{T}$ with the same notations as in Theorem 4.

**Proposition 1.** *The functions families $\left(H_1^\delta\right)_{\delta>0}$, $\left(H_j^\delta\right)_{\delta>0}$, $\left(H_{j^2}^\delta\right)_{\delta>0}$ are* pre-compact *on the closure of $\mathcal{T}$ with respect to the uniform norm, that is, each sequence admits a uniformly converging subsequence.*

**Proposition 2.** *If $\tilde{h_1}, \tilde{h_j}, \tilde{h_{j^2}}$ are accumulation points given by the previous lemma, then the linear combination*

$$\tilde{g} = \sum_{\mu \in \mathcal{C}_3} \left(\frac{1}{3} + \frac{2}{3}\mu\right) \tilde{h_\mu},$$

*once extended to the closure of $\mathcal{T}$, is a homeomorphism from the boundary of $\mathcal{T}$ to the boundary of $\Delta$ and maps the vertices of $\mathcal{T}$ $a_1, a_j, a_{j^2}$ to the corresponding vertices $1, \frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}$ of $\Delta$.*

**Proposition 3.** *With the same notations as in previous proposition, $\tilde{h_1} + \tilde{h_j} + \tilde{h_{j^2}}$ and $\tilde{g}$ are holomorphic on the interior of $\mathcal{T}$.*

Let us recall the following theorem (see [2] for instance):

**Theorem 5** (Darboux-Picard)**.** *Let $\Omega$ and $\Omega'$ be two complex domains bounded by Jordan curves. If $\Phi : \Omega \to \Omega'$ is a holomorphic function that extends continuously to the closure of $\Omega$ and maps the boundary of $\Omega$ homeomorphically on the boundary of $\Omega'$, then $\Phi$ is a conformal mapping between $\Omega$ and $\Omega'$.*

With the notation introduced in the propositions we have that the sum $\tilde{h_1} + \tilde{h_j} + \tilde{h_{j^2}}$ is constant (since it is a real-valued holomorphic function), and by Theorem 5, $\tilde{g}$ is a conformal mapping from $\mathcal{T}$ to $\Delta$.

This conformal mapping is unique, so it is in fact equal to the function $g$ defined in Theorem 4. Therefore for each $\mu \in \mathcal{C}_3$, $\tilde{h_\mu}$ is unique, so we denote it by $h_\mu$, as in the Theorem 3. By uniqueness of the accumulation point as $\delta$ tends to 0, we have that the function $H_\mu^\delta$ actually converges to $h_\mu$ for each $\mu \in \mathcal{C}_3$.

So it suffices to show Propositions 1, 2, 3. This will be done in the next paragraphs.

# 2 Standard properties

We now give some standard results that will be used for the proof of the propositions of previous section that are specific to triangular lattices.

## 2.1 Self-duality

Self-duality is a property that appears on the triangular lattice in various equivalent forms, that states essentially that if there is no white path in some direction, then there is a black path in the "orthogonal" direction. Although easy to figure out, this graph-theoretic result is difficult to prove (see [5], for instance). We give two equivalent forms of this property that we will need in this report, the first having already be mentioned in the previous section.

**Proposition 4** (Self-duality, rectangular form)**.** *Let $\mathcal{R}$ be a discretised conformal rectangle with vertices a, b, c, d in counterclockwise order. Then if there is no white path separating $a_\delta$ and $b_\delta$ from $c_\delta$ and $d_\delta$, then there is a black path separating $a_\delta$ and $d_\delta$ from $b_\delta$ and $c_\delta$. Both events cannot occur simultaneously, so they form a partition of the probability space.*

**Proposition 5** (Self-duality, radial form)**.** *Let $\Omega$ be a discretised domain and $z$ be a vertex in $\Omega$. Then if there is no white annulus (closed simple path) surrounding $z$, then there is a black path joining $z$ to the boundary of $\Omega$. Conversely, if there is no such black path, then there exists a white annulus surrounding $z$.*

### 2.2 Russo-Seymour-Welsh theorem

A very important tool for the estimation of crossing probabilities is the Russo-Seymour-Welsh theorem which we will use in two forms. As usual we consider independent percolation on the equilateral triangular lattice with parameter $p = \frac{1}{2}$.

**Theorem 6** (Russo-Seymour-Welsh, rectangular form)**.** *Let $\mathcal{R}$ be a conformal rectangle with vertices $a$, $b$, $c$, $d$ in counterclockwise order. There exists constants $\alpha > 0$ and $\beta < 1$ such that the probability $C_\delta(\mathcal{R}, a, b, c, d)$ of a white (or black) crossing separating $a_\delta$ and $b_\delta$ from $c_\delta$ and $d_\delta$ is bounded uniformly with respect to the mesh $\delta$:*

$$\alpha \leq C_\delta(\mathcal{R}, a, b, c, d) \leq \beta \quad \forall \delta > 0$$

**Theorem 7** (Russo-Seymour-Welsh, radial form)**.** *Consider percolation in a disc $\mathcal{D}$ with center $z$ of radius $\alpha > 0$, with $z$ and $\alpha$ fixed. There exists positive constants $C$ and $\epsilon$ such that for any circle $\Gamma$ centered in $z$ of radius $r < \alpha$ the probability that there is a white (or black) path joining $\Gamma$ to the boundary of $\mathcal{D}$ is smaller that $Cr^\epsilon$, for all mesh $\delta > 0$.*

## 3 Compactness

In order to show the precompactness of the functions families $\left(H_\mu^\delta\right)_{\delta>0}$ asserted in Proposition 1, we shall suppose in this section that $\mu = 1$ (the other cases $\mu = j, j^2$ are symmetric) and write $H_\delta$ for $H_1^\delta$.

We show that each sequence $(H_{\delta_n})_{n\geq 0}$ with $\delta_n$ tending to 0 admits an accumulation point. For this purpose we will use the following lemma:

**Lemma 1.** *There exist positive constants $C$ and $\epsilon$ such that, for $z_1, z_2$ vertices of the triangular discretisation of $\mathcal{T}$ (equivalently, centers of the hexagonal faces), we have, for all mesh $\delta > 0$,*

$$|H_\delta(z_1) - H_\delta(z_2)| \leq C \cdot \ell(z_1, z_2)^\epsilon,$$

*where $\ell(z_1, z_2)$, the* connectivity function*, is the length of the shortest path joining $z_1$ and $z_2$ in the of the triangle $\mathcal{T}$.*

We obtain the desired accumulation point by interpolation: for $\delta > 0$, we denote by $\tilde{H}_\delta$ an interpolation of $H_\delta$ defined on the closure of the triangle $\mathcal{T}$.

From the lemma, we obtain that (a suitably choosen) family of interpolations $\left(\tilde{H}_\delta\right)_{\delta>0}$ is Hölder with respect to the distance $\ell$, which is equivalent to the euclidean metric ($\ell(z_1, z_2) \to 0$ uniformly as $|z_1 - z_2| \to 0$) since $\mathcal{T}$ is bounded by a Jordan curve. This implies that the family $\tilde{H}_\delta$ is *equicontinuous*:

$$\forall \epsilon > 0 \; \exists \delta > 0 : \; |x - y| \le \delta \Longrightarrow |f_i(x) - f_i(y)| \le \epsilon \; \forall i \in I.$$

We may now apply the following classical compactness theorem (see [8])

**Theorem 8** (Ascoli-Arzela)**.** *A bounded equicontinuous family of functions defined on a compact set is precompact with respect to the uniform norm.*

So we obtain for each sequence $\delta_n$ tending to 0 an accumulation point $\tilde{h}$ of the sequence $\tilde{H_{\delta_n}}$. But $\tilde{h}$ is also an accumulation point of the sequence $H_{\delta_n}$, since the difference between the original functions $H_\delta$ and their interpolation $\tilde{H}_\delta$ tends to 0 as the mesh goes to 0, and we obtain the desired result.

To complete the proof, it remains to show Lemma 1, which is done mainly by using Russo-Seymour-Welsh estimates.

#### 3.0.1 Proof of Lemma 1

As in the formulation of Lemma 1, let $z_1, z_2$ be the two vertices inside the triangulation. By elementary partitioning we obtain (recall that $H_\delta(z)$ is the probability of the event $Q_1^\delta$ that there exists a white path separating $a_1$ and $z$ from $a_j$ and $a_{j^2}$)

$$\left| H_1^\delta(z_1) - H_1^\delta(z_2) \right| = \left| \mathbf{P}\left(Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)\right) - \mathbf{P}\left(Q_1^\delta(z_2) \setminus Q_1^\delta(z_1)\right) \right|.$$

So by symmetry of the latter expression it is sufficient to show that there are positive constants $C'$ and $\epsilon$ such that

$$\mathbf{P}\left(Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)\right) \le C' \ell(z_1, z_2)^\epsilon.$$

To use Russo-Seymour-Welsh theorem (Theorem 7) we want to see that the event $Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)$ implies a connection between two circles, one of *macroscopic size* (that is, depending on the shape of the domain of $\mathcal{T}$ only) and one of *microscopic size* (of order $\ell(z_1, z_2)$). The size informations are given by the following geometrical considerations.

- We first remark that there exists a positive constant $\alpha > 0$ (depending on the $\mathcal{T}$ only) such that each point of $\mathcal{T}$ is at (euclidean) distance of at least $\alpha$ to one of its three sides $A_1$, $A_j$ and $A_{j^2}$.
- If $z_1$ and $z_2$ are close enough (which we may suppose to show the Hölder condition since $H_\delta$ is bounded), we can suppose that they are both contained inside a (microscopic) circle of radius $\ell(z_1, z_2)$ which is at (macroscopic) distance at least $\frac{\alpha}{2}$ to one of the sides $\mathcal{T}$.

It remains to show that the event $Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)$ implies the occurence of a connection by a monochromatic path from the microscopic circle of radius $\ell(z_1, z_2)$ to the farthest side of the triangle (a fortiori to a circle of radius $\frac{\alpha}{2}$), which is by Russo-Seymour-Welsh theorem an event of probability less that $C'\ell(z_1, z_2)^\epsilon$ for some positive constants $C'$ and $\epsilon$ and this yields the desired result.

In fact, if $Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)$ happens, then the microscopic circle is connected by a monochromatic path to each of the three sides of the triangle (see Figure 6).

- It is connected by a white path to $A_j$ and $A_{j^2}$, since there is a white path that separates $z_1$ and $z_2$: $z_1$ is separated from $a_j$ and $a_{j^2}$ while $z_2$ is not, so the separation must run between them.
- There is a black path that connects the circle to $A_1$, since otherwise, by self-duality, there would be a white path separating it from $A_1$, which would imply the occurence of $Q_1^\delta(z_2)$.

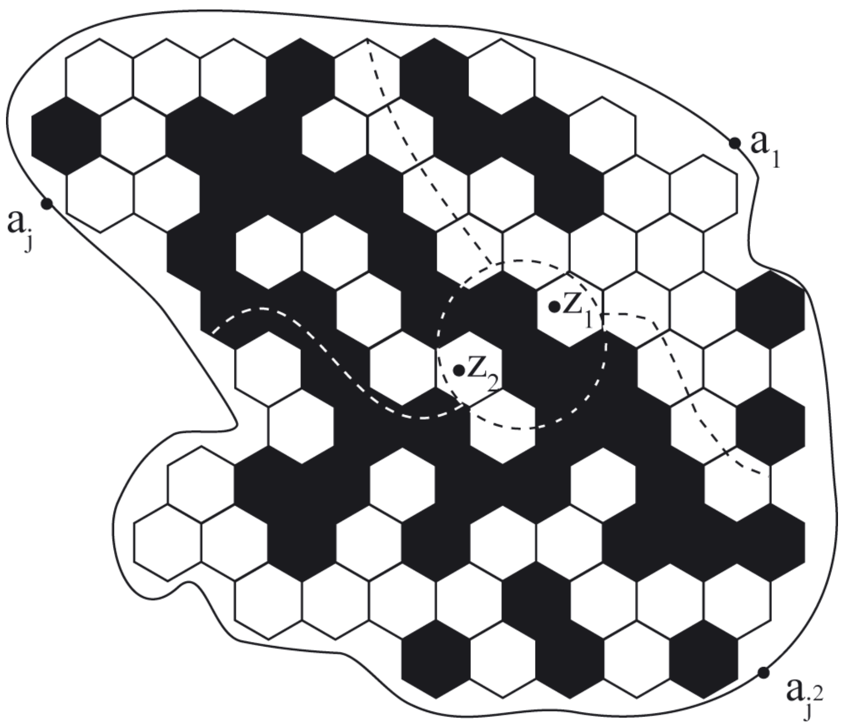


Figure 7: Occurence of $Q_1^\delta(z_1) \setminus Q_1^\delta(z_2)$ and monochromatic connections to the three sides of $\mathcal{T}$

# 4 Boundary behaviour

Let $\tilde{h_1}, \tilde{h_j}, \tilde{h_{j^2}}$ be accumulation points of the functions $H_1^\delta, H_j^\delta, H_{j^2}^\delta$ as $\delta \to 0$. The goal of this section is to prove Proposition 2: the linear combination $\tilde{g} = \sum_{\mu \in \mathcal{C}_3} \left(\frac{1}{3} + \frac{2}{3}\mu\right) \tilde{h_\mu}$ maps the boundary of our conformal triangle $\mathcal{T}$ homeomorphically on the boundary of the equilateral triangle $\Delta$ and sends the points $a_1$, $a_j$, $a_{j^2}$ on the corresponding vertices 1, $\frac{i}{\sqrt{3}}$, $-\frac{i}{\sqrt{3}}$ of $\Delta$, respectively. This assertion is the first step to show that $\tilde{g}$ is in fact a conformal mapping between $\mathcal{T}$ and $\Delta$.

We will prove the following claims, which imply Proposition 2 (bijectivity comes from the boundary conditions, and since the boundary of $\mathcal{T}$ is compact, $\tilde{g}$ is a homeomorphism on it):

1. The vertices of $\mathcal{T}$ are mapped on the vertices of $\Delta$: $\tilde{g}(a_1) = 1$, $\tilde{g}(a_j) = \frac{i}{\sqrt{3}}$, $\tilde{g}(a_{j^2}) = -\frac{i}{\sqrt{3}}$.
2. The sides of $\mathcal{T}$ are mapped on the sides of $\Delta$: $\tilde{g}(A_1) \subset \left[\frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}\right]$, $\tilde{g}(A_j) \subset \left[1, -\frac{i}{\sqrt{3}}\right]$, $\tilde{g}(A_{j^2}) \subset \left[1, \frac{i}{\sqrt{3}}\right]$.
3. $\tilde{g}$ is injective on the boundary of $\mathcal{T}$.

Of course, by symmetry, it is sufficient to show the first claim for $a_1$ and the second one for $A_1$.

*Proof of Claim 1.* We show that $\lim_{\delta\to 0} H_1^\delta(a_1^\delta) = 1$ and $\lim_{\delta\to 0} H_j^\delta(a_1^\delta) = \lim_{\delta\to 0} H_{j^2}^\delta(a_1^\delta) = 0$ where $a_1^\delta$ is the discretisation of the vertex $a_1$ at mesh $\delta$.

These facts come from self-duality arguments: if $Q_1^\delta(a_1^\delta)$ (recall that $H_1^\delta = \mathbf{P}\left(Q_1^\delta\right)$) does not occur, then there exists a black path that joins $a_1^\delta$ to the opposite arc $A_1$, implying the connection of a microscopic circle (of order $\delta$) to a macroscopic one (depending on $\mathcal{T}$ only).

By Russo-Seymour-Welsh theorem, the probability of non-occurence of $Q_1^\delta(z_\delta)$ tends to 0 as $\delta \to 0$, and therefore $\lim_{\delta\to 0} H_1^\delta(z_\delta) = 1$. The results for $H_j^\delta$ and $H_{j^2}^\delta$ are proved exactly the same way (the corresponding events imply occurence of a monochromatic connection of small probability).

*Proof of Claim 2.* First we prove that $\lim_{\delta\to 0} H_1^\delta(z_\delta) = 0$ for the discretisation $z_\delta$ of a point on the arc $A_1$. Again, this follows from Russo-Seymour-Welsh theorem: the corresponding event implies that a small circle around $z_\delta$ is connected by a white path to the arcs $A_j$ and $A_{j^2}$, at least one of them being at macroscopic distance of $z_\delta$, and we conclude as in the previous assertion.

So it remains to see that $\tilde{h_j} + \tilde{h_{j^2}} = 1$ on $A_1$, so that $\tilde{g}$ become a convex combination of $\frac{1}{3} + \frac{2}{3}j$ and $\frac{1}{3} - \frac{2}{3}j$. This comes from the fact that the events $Q_j^\delta$ and $Q_{j^2}^\delta$ partition the probability space, by self-duality.

*Proof of Claim 3.* If we show that the function $\tilde{h_1}$ is strictly monotonic on the arc $A_j$, then by symmetry it is also on the other arc $A_{j^2}$, and the functions $\tilde{h_j}$ and $\tilde{h_{j^2}}$ satisfy the same property for the corresponding arcs. Since on each arc one of the three functions vanishes and the two others are one-to-one, we obtain by linear independance that $\tilde{g}$ is injective on each arc, and by Claim 2, $\tilde{g}$ is therefore one-to-one. To see why $\tilde{h_1}$ is strictly increasing on $A_j$, take two points $z_1$ and $z_2$ on $A_j$ such that we have in that order $a_1, z_1, z_2, a_j$. We want to show that $\tilde{h_1}(z_1) > \tilde{h_1}(z_2)$. This comes from the fact that the probability that there is white path that separates $z_1$ and $a_1$ from $a_j$ and $a_{j^2}$ and that this is not the case for $z_2$, is bounded away from zero by Russo-Seymour-Welsh theorem. By constructing appropriate disjoint "tunnels" (conformal rectangles), one can ensure that with positive probability there exists a white path separating $z_1$ and $a_1$ from $z_2$, $a_j$ and $a_{j^2}$ and simultaneously a black path between the white path and $z_2$ that joins $A_j$ to $A_1$ (the existence of this black path ensures that the event $Q_1^\delta(z_2)$ does not occur).

# 5 Analyticity

The last step in order to complete the proof of Smirnov's Theorem (Theorem 5) is to show the analyticity of the functions $\tilde{h_1} + \tilde{h_j} + \tilde{h_{j^2}}$ and $\tilde{g} = \sum_{\mu \in \mathcal{C}_3} \left(\frac{1}{3} + \frac{2}{3}\mu\right) \tilde{h_\mu}$ defined in the first section. In the following paragraphs we show that the function $h$ defined by $h = \tilde{h_1} + j\tilde{h_j} + j^2\tilde{h_{j^2}}$ is analytic ; the proof for $\tilde{h_1} + \tilde{h_j} + \tilde{h_{j^2}}$ is similar and we omit it here. Eventually, as $g$ is a linear combination of these two functions, we obtain the claimed result.

Let us denote by $\delta_n$ a sequence tending to 0 such that the sequences $H_\mu^{\delta_n}$ converge to $\tilde{h_\mu}$ for $\mu \in \mathcal{C}_3$. For shorter notation, we write $H_\delta$ for the sum $H_1^\delta + jH_j^\delta + j^2 H_{j^2}^\delta$ (this is not the function defined in Section 2). So we have that $H_{\delta_n}$ tends to $h$ as $n$ goes to infinity.

In order to show that $h$ is holomorphic it suffices, by the following theorem, to show that the integral of $h$ along any simple smooth curve contained in the conformal triangle $\mathcal{T}$ is equal to zero.

**Theorem 9** (Morera's condition)**.** *Let $\Omega$ be a simply connected domain of the complex plane and let $f : \Omega \to \mathbb{C}$ be a continuous function. Then $f$ is holomorphic if and only if for every simple closed smooth curve $\gamma$ contained in $\Omega$ the integral of $f$ along $\gamma$ is 0.*

So let $\gamma$ be simple smooth curve contained in $\mathcal{T}$. For a mesh $\delta > 0$, let us denote by $\mathcal{T}_\delta$ the discretisation of $\mathcal{T}$ in regular hexagons of size $\delta$ and by $\gamma_\delta$ the discretisation of $\gamma$ constructed as a closed path (in the graph-theoretical sense) in $\mathcal{T}_\delta$ such that $\gamma_\delta$ tends to $\gamma$ as $\delta$ tends to 0 (with respect to the Hausdorff distance).

Now we construct an approximation of the integral $\oint_\gamma h(z)\,\mathrm{d}z$ in the following way. For an oriented edge $\vec{e} = (x, y)$ in the graph $\mathcal{T}_\delta$ ($x$ and $y$ being its initial and terminal vertices respectively), define the following quantities:

$$H_\delta(\vec{e}) = \frac{H_\delta(x) + H_\delta(y)}{2}, \quad \partial_{\vec{e}} H_\delta = H_\delta(y) - H_\delta(x).$$

Let $\vec{\gamma_\delta}$ denotes the set of the edges of $\gamma_\delta$ oriented in the counterclockwise orientation. We define the discrete integral of $H_\delta$ along $\gamma_\delta$ as

$$I_\delta(\gamma) = \sum_{\vec{e} \in \vec{\gamma_\delta}} \vec{e} H_\delta\left(\vec{e}\right),$$

where $\vec{e}$ before $H_\delta$ is seen as the complex number $y - x$. This sum approximates a Riemann integral of $H_\delta$, so that we have

$$I_{\delta_n}(\gamma) \to \oint_\gamma h(z)\,\mathrm{d}z \quad \text{as } n \to \infty.$$

Let us denote by $\mathrm{Int}\left(\gamma_\delta\right)$ the set of the hexagonal faces contained in the domain bounded by $\gamma_\delta$ and for such a face $f$ by $\vec{\partial f}$ its six vertices oriented in counterclockwise orientation. So we obtain

$$I_\delta(\gamma) = \sum_{f \in \mathrm{Int}(\gamma_\delta)} \sum_{\vec{e} \in \vec{\partial f}} \vec{e} H_\delta\left(\vec{e}\right).$$

since the terms appearing in edges that are not on the boundary $\gamma_\delta$ appear twice with opposite signs and therefore cancel (see Figure 8).

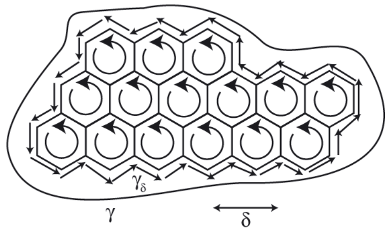


Figure 8: The contributions of the edges that are not on the boundary cancel

An elementary reordering of the terms give that for each hexagonal face $f \in \vec{\partial f}$ with edges $(x_0, x_1), (x_1, x_2), \ldots, (x_5, x_0)$ and with center $c(f)$

$$\sum_{\vec{e} \in \vec{\partial f}} \vec{e} H_\delta(\vec{e}) = \sum_{k=0}^{5} \left( \frac{x_k + x_{k+1}}{2} - c(f) \right) \left(H(x_{k+1}) - H(x_k)\right),$$

the index $k$ being taken modulo 6.

For vertices not in the boundary $\gamma_\delta$, the term $\frac{x_k+x_{k+1}}{2}\left(H(x_{k+1})-H(x_k)\right)$ appears twice with opposite signs and cancels also, so only the terms with the factor $c(f)$ remain. A term of the form $H(x_{k+1})-H(x_k)$ becomes a factor of the difference between two faces so it is a *dual edge* of $(x_k, x_{k+1})$ obtained by rotating the said edge by an angle of $\frac{\pi}{2}$. We denote by $\vec{e}^*$ the dual edge of an oriented edge $\vec{e}$.

On the other hand, the terms in the boundary count for little: we have that the number of edges appearing in $\gamma_\delta$ is of order $\frac{1}{\delta}$ while the term $\frac{x_k+x_{k+1}}{2}-c(f)$ is of order $\delta$ and the difference is of order $\delta^\epsilon$ since we proved in the previous section that the functions are Hölder. So the contribution of the boundary is of order $\delta^\epsilon$.

Putting together these two remarks we eventually obtain

$$I_\delta(\gamma)=\frac{1}{2}\sum \vec{e}^*\partial_{\vec{e}}H_\delta+\mathrm{o}\left(\delta^\epsilon\right),$$

where the sum is taken over all the positively oriented edges of the hexagonal faces lying in the strict interior of $\gamma_\delta$.

We have that $\partial_{\vec{e}}H_\delta=\partial_{\vec{e}}H_1^\delta+j\partial_{\vec{e}}H_j^\delta+j^2\partial_{\vec{e}}H_{j^2}^\delta$, the terms in the right hand side being defined the same way as $\partial_{\vec{e}}H_\delta$. For an edge $\vec{e}=(x,y)$ we define $P_1^\delta(\vec{e})$ as the probability of the event $Q_1^\delta(y)-Q_1^\delta(z)$ (and we define similarly $P_j^\delta, P_{j^2}^\delta$). As we have $\partial_{\vec{e}}H_1^\delta=P_1^\delta(\vec{e})-P_1^\delta(-\vec{e})$ by elementary partitioning, we obtain

$$I_\delta(\gamma)=\sum \vec{e}^*\left(P_1^\delta(\vec{e})+jP_j^\delta(\vec{e})+j^2P_{j^2}^\delta(\vec{e})\right)+\mathrm{o}\left(\delta^\epsilon\right)$$

Using the following lemma will enable us to complete the proof. For an edge in the hexagonal lattice $\vec{e}=(x,y)$, we denote by $j\vec{e}$ (respectively by $j^2\vec{e}$) the edge obtained by rotating $\vec{e}$ by an angle of $\frac{2\pi}{3}$ (respectively $\frac{4\pi}{3}$) around its intial vertex $x$.

**Lemma 2** (Smirnov). *For every edge $\vec{e}$ in the subgraph delimited by $\gamma_\delta$ we have the following identity*

$$P_1^\delta(\vec{e})=P_j^\delta(j\vec{e})=P_{j^2}^\delta(j^2\vec{e}).$$

Using Lemma 2, we can rearrange the sum $I_\delta(\gamma)$:

$$\begin{aligned}
I_\delta(\gamma) &= \sum \vec{e}^*\left(P_1^\delta(\vec{e})+jP_1^\delta(j^2\vec{e})+j^2P_1^\delta(j\vec{e})\right)+\mathrm{o}\left(\delta^\epsilon\right)\\
&= \sum\left(\vec{e}^*+j(j\vec{e})^*+j^2(j^2\vec{e})^*\right)P_1^\delta(\vec{e})+\mathrm{o}\left(\delta^\epsilon\right).
\end{aligned}$$

As $\vec{e}^*+j(j\vec{e})^*+j^2(j^2\vec{e})^*=0$, we have that the integral vanishes as $\delta$ tends to 0, concluding the proof of Proposition 3; so the only part remaining to be shown is Lemma 2.

## 5.1 Proof of Lemma 2

We only show that $P_1^\delta(\vec{e}) = P_j^\delta(j\vec{e})$, the other part of the identity being symmetric. So let us write $\vec{e} = (x, y)$ and denote by $x^*$ the triangular face in the triangular discretization $\mathcal{T}_\delta$ which has $x$ as its center.

The occurence of the event $Q$ defined by $Q = Q_1^\delta(y) - Q_1^\delta(x)$ (recall that $P_1^\delta(\vec{e})$ is the probability of $Q$) implies the existence of three monochromatic connections with the sides of the conformal triangle $\mathcal{T}$, as seen in Lemma 1 (Section 2). More precisely, there is a white path separating $A_j$ and $A_{j^2}$ that passes between $x$ and $y$, such that the triangular face $x^*$ has two vertices on it : denote by $v_j$ the vertex that is the closer to $A_j$ (in the sense of the white path) and by $v_{j^2}$ the one that is closer to $A_{j^2}$. Denote by $v_1$ the third vertex of $x^*$; by self-duality $v_1$ is connnected to $A_1$ by a black path (since otherwise $Q_1^\delta(x)$ would occur).

So, if we denote by $Q_{bww}$ the event that there exists a black connection (first letter of the index) between $v_1$ and $A_1$, a white connection between $v_j$ and $A_j$ (second letter of the index) and a white connection between $v_{j^2}$ and $A_{j^2}$ (third letter of the index), we have that the occurence of $Q$ implies the one of $Q_{bww}$. Conversely, $Q_{bww}$ implies $Q$, so $Q = Q_{bww}$. Similarly, we have that $P_j^\delta(j\vec{e})$ is the probability of $Q_{wbw}$.

As we consider an independent percolation process with parameter $p = \frac{1}{2}$, to show that $P_1^\delta(\vec{e}) = P_j^\delta(j\vec{e})$, it suffices to see that the cardinality of $Q_{bww}$ (the number of configurations (colourings) such that $Q_{bww}$ occurs) is equal to the cardinality of $Q_{bwb}$, or equivalently to construct a bijection between the configurations of these two events. By flipping the colours we have a bijection between $Q_{bwb}$ and $Q_{wbw}$, so it suffices to construct a bijective map $\Psi : Q_{bww} \to Q_{bwb}$.

So, let $\omega$ be a configuration of $Q_{bww}$, and let $B_1$ be the counterclockise-most extremal black path joining $v_1$ to $A_1$ and $W_j$ the clockwise-most extremal white path joining $v_j$ to $A_j$ (it is not difficult to see that it is well-defined). The concatenation of $B_1$ and $W_j$ is path that joins $A_1$ and $A_j$.

We define $\Psi(\omega)$ by flipping the colours of the hexagonal faces in the subgraph of $\mathcal{T}_\delta$ strictly delimited by the concatenation of $B_1$ and $W_j$ which is on the side of $a_1$ and $a_j$. So $B_1$ and $W_j$ are left unchanged, while we have now that $v_{j^2}$ is joined to $A_{j^2}$ by a black path. So $\Phi(\omega)$ is now a configuration of $Q_{bwb}$ and we have constructed the mapping $\Phi$ as claimed. By flipping again the colours, we can construct similarly the inverse of $\Psi$ so it is bijective and this concludes the proof: we have

$$P_1^\delta(\vec{e}) = \mathbf{P}(Q_{bwb}) = \mathbf{P}(Q_{bwb}) = \mathbf{P}(Q_{wbw}) = P_j^\delta(j\vec{e}).$$

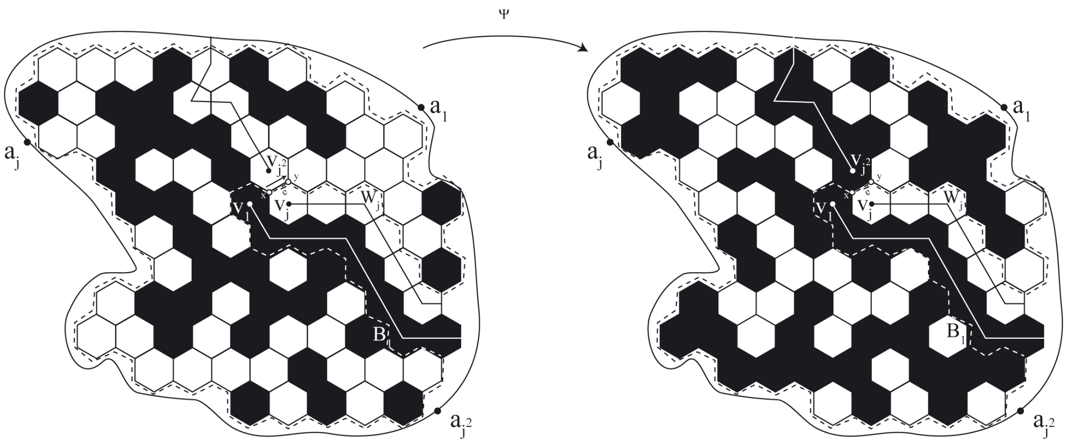


Figure 9: Flipping the colours

## 6 Cardy's formula and $SLE_6$

As we have seen in the first section, Smirnov's theorem not only states the conformal invariance of the limit of crossing probabilities, but also gives, through Carleson's reformulation (Corollary 1), an explicit formula for calculating these probabilities: from this reformulation, we obtain immediatly that on the equilateral triangle with vertices $a = 1, b = \frac{i}{\sqrt{3}}, c = -\frac{i}{\sqrt{3}}$, we have, for every point $d$ on the segment $[a, c]$, that the probability that a white path separates $a$ and $d$ from $b$ and $c$ (in fact we should talk about the respective discretization of $a, b, c, d$, but we will make this slight abuse of notation from now) tends to $\mathrm{Re}(d)$ as the mesh $\delta$ tends to 0.

Historically, Cardy, who conjectured conformal invariance for several statistical physics models (not only percolation, but also for instance Ising model) at critical probability, gave an explicit formula for the unit disc $\mathbb{D} = D(0, 1)$, which can be proved in the particular case of the equilateral triangular lattice using Smirnov's theorem.

Let $a, b, c, d$ be four points on the boundary of $\mathbb{D}$, counterclockwise. We denote by $u$ the *cross-ratio* of $a, b, c, d$ defined by $u = \frac{(d-c)(b-a)}{(c-a)(d-b)}$, which is a real number between 0 and 1.

**Proposition 6** (Cardy's formula)**.** *The probability $C_\delta(\mathbb{D}, a, b, c, d)$ that a white path separates $a$ and $d$ from $b$ and $c$ tends to a function $f$ of the cross-ratio $u$, where $f$ satisfies the hypergeometric differential equation*

$$u(1-u)f''(z) + \frac{2}{3}(1-2u)f'(u) = 0,$$

*with the obvious boundary conditions $f(0) = 0$ and $f(1) = 1$. The solution* *of this equation is given by the hypergeometric function*

$$\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}\int_0^u (1-x)^{-\frac{2}{3}}x^{-\frac{2}{3}}\,\mathrm{d}x.$$

Notice that the fact that the limit is a function of $u$ only makes sense, since it is conformally invariant and conformal mappings of $\mathbb{D}$ on itself are homographies and preserve the cross-ratio (see [8], for instance).

We could derive the solution of the equation by mapping the equilateral triangle conformally (such a mapping is a hypergeometric function) on the unit disc and translate Carleson's formulation in terms of the cross-ratio, but this does not tell from where the differential equation comes, and we will try to show a way to obtain it in the following paragraphs.

By conformal invariance, is sufficient to derive the formula for specific points on the boundary of $\mathbb{D}$, so let us fix $b=-1, c=-i, d=1$ and let $a$ move freely on $\mathrm{arc}(d,b)$ (the arc joining $d$ to $b$ counterclockwise), so that $u$ runs through the whole open interval $]0,1[$. Consider the usual discretization of $\mathbb{D}$ and the points at mesh $\delta>0$ and the associated percolation process.

For any configuration (colouring) $\omega$, we define an *exploration process* on the vertices of the hexagonal faces (so we consider the dual of the initial graph $\mathcal{G}_\delta$) in $\mathbb{D}$ in the following manner.

1. We first add to the discretization of $\mathbb{D}$ the hexagons of $\mathcal{G}_\delta$ which lie on the external boundary on $\mathrm{arc}(a,c)$ (we only took the hexagons inside $\mathbb{D}$ for the percolation process, so we add one layer of hexagons to the discretization of $\mathbb{D}$) and we colour in black all the ones on $\mathrm{arc}(b,c)$, and in white all the ones on $\mathrm{arc}(a,b)$.

2. We start from the vertex $x_0$ that is the nearest from $b$ and consider the edge $v_1=(x_0,x_1)$ that does not belong to the discretized boundary of $\mathbb{D}$ (we suppose $\delta$ sufficiently small so that it is possible).

3. The the process runs step by step through the graph: at step $n$, we define $e_{n+1}$ and $x_{n+1}$ as follows. Consider the two edges starting from $v_n$ and distinct from $e_n$. Call $b_n$ the one which is "on the left" and $w_n$ the one which is "on the right" (we have in counterclockwise order: $w_n, b_n, e_n$). Denote by $\xi_n$ the hexagonal face to which $b_n$ and $w_n$ are adjacent. Then:

   - If $\xi_n$ is black, take $e_{n+1}=(v_n,v_{n+1})=b_n$.
   - If $\xi_n$ is white, take $e_{n+1}=(v_n,v_{n+1})=w_n$.

4. Stop when the process reaches $\mathrm{arc}(c,d)$ or $\mathrm{arc}(d,a)$.

The exploration process has on its left a layer of white hexagons and on its right a layer of black hexagons. So if it reaches $\mathrm{arc}(d,a)$, then obviously there can be no white path separating $a$ and $d$ from $b$ and $c$. On the contrary, if it reaches $\mathrm{arc}(c, d)$, by self-duality there is a white path separating $a$ and $d$ from $b$ and $c$ (otherwise it would have reached $\mathrm{arc}(d, a)$). The initial colouring of $\mathrm{arc}(b, c)$ and $\mathrm{arc}(a, b)$ ensures termination of the process.

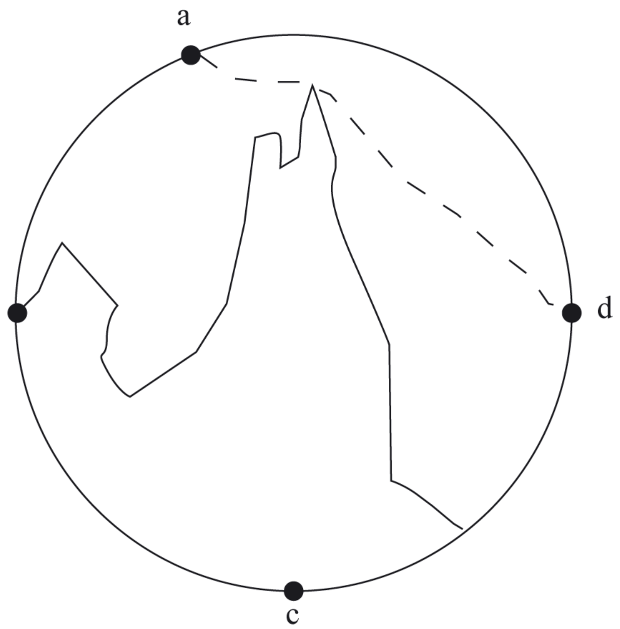


Figure 10: The exploration process (solid) avoids the white path (dashed)

It has been proved using an idea of Smirnov (see [3]) that the exploration process converges to a continuous limit as the mesh tends to 0, called chordal $SLE$ (for *Schramm-Loewner Evolution*) with parameter $\kappa = 6$, or (chordal) $SLE_6$. We briefly define the family of the $SLE_\kappa$ processes in the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. For more details see [6], p.147.

The $SLE_\kappa$ process with parameter $\kappa \geq 0$ on the plane $\mathbb{D}$ is the random collection of conformal maps $g_t$ obtained by solving the initial value problem

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z \quad (z \in \mathbb{H}),$$

where $B_t$ is a standard one-dimensional Brownian motion starting at the origin. Each $g_t$ is a conformal mapping from a domain $H_t \subset \mathbb{H}$ to $\mathbb{H}$.

When we say that $\gamma$ is a $SLE$ curve, we mean by that that the domain $H_t$ of $g_t$ is the set of points in $\mathbb{H}$ that are connected to the point at infinity (denoted by $\infty$) by a path that does not cross $\gamma$. We say that $\gamma$ *generates* the $SLE$. We have that $SLE$ is generated by a path with probability one (see [6], Theorem 6.3, p. 148).

For a point $z$ in the closure of $\mathbb{H}$, we define the hitting time $T_z$ as the first time that $z$ leaves the closure of the domain of $g_t$. In other words, it is the time when $z$ is "swallowed" by the $SLE$.

We map conformally $\bar{\mathbb{H}} \cup \{\infty\}$ on the unit disc $\bar{\mathbb{D}}$ by the application $\phi : \bar{\mathbb{H}} \cup \{\infty\} \to \bar{\mathbb{D}}$ defined by

$$\phi(z) = \frac{z-i}{z+i}.$$

We have $\phi(0) = b$, $\phi(1) = c$, $\phi(\infty) = d$. As one can easily check, $\phi$ is an homeomorphism between the negative real line $\{-y : y > 0\}$ and $\mathrm{arc}(d, b)$, so for any $a$ on $\mathrm{arc}(d, b)$, we find $y > 0$ such that $\phi(-y) = a$. We define $SLE$ on the disc $\mathbb{D}$ starting at $b$ as the image by the conformal mapping $\phi$ of $SLE$ on $\mathbb{H}$.

As the limit of the exploration process defined above is a $SLE_6$ curve starting from $b$ we have that the event that the process hits $\mathrm{arc}(c, d)$ (instead of $\mathrm{arc}(d, a)$) corresponds in the limit to the event $T_c < T_a$ on $\mathbb{D}$ (the hitting time being defined similarly on $\mathbb{D}$ as on $\mathbb{H}$) which can be translated into the event $T_{-y} < T_1$ for $SLE_6$ on $\mathbb{H}$.

So we summarize the remarks above in the following proposition.

**Proposition 7.** *The limit of the probability $C_\delta(\mathbb{D}, a, b, c, d)$ (as $\delta \to 0$) is equal to the probability that a $SLE_6$ path on $\mathbb{D}$ hits* $\mathrm{arc}(c, d)$ *before* $\mathrm{arc}(d, a)$*, or equivalently that $T_{-y} < T_1$ on $\mathbb{H}$.*

The probability of the event $T_{-y} < T_1$ is described by the following proposition (see [6], Proposition 6.33, p.165), which follows from Itô's formula.

**Proposition 8.** *If $y > 0$ and $\gamma$ is an $SLE_\kappa$ curve with $\kappa > 4$ then*

$$\mathbf{P}(T_{-y} > T_1) = \psi\left(\frac{y}{y+1}\right),$$

*where $\psi$ satisfies the hypergeometric equation:*

$$u(1-u)\psi''(u) + (2a - 4au)\,\psi'(u) = 0,$$

*with $a = \frac{2}{\kappa}$.*

Notice that the proposition gives an equation for the probability

$$\mathbf{P}(T_{-y} > T_1) = 1 - \mathbf{P}(T_{-y} < T_1),$$

but by linearity, we have of course that $\mathbf{P}(T_{-y} < T_1)$ satisfies the same equation.

Eventually, taking $\kappa = 6$ in the proposition yields the differential equation of Proposition 6, since we have

$$\lim_{\delta \to 0} C_\delta(\mathbb{D}, a, b, c, d) = \mathbf{P}(T_c < T_a) = \mathbf{P}(T_{-y} < T_1),$$

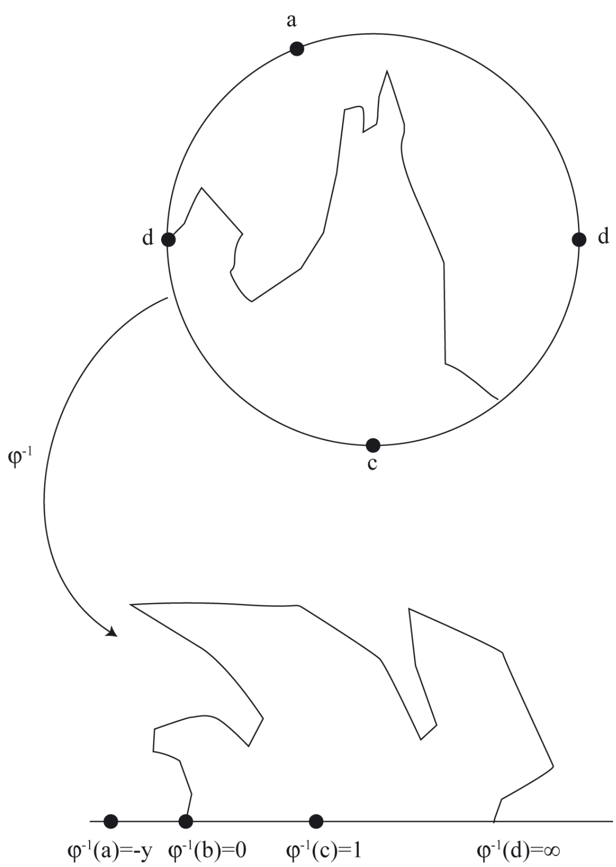


Figure 11: Taking back $SLE$ in the half-plane

and using the fact that the cross-ratio of the points $a, b, c, d$ is in fact equal to $\frac{y}{y+1}$.

*Many thanks to Reda Messikh and Jiří Černý for introducing me to this fascinating suject and for their help, advices and the numerous hours they spent for me!*

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