Critical Percolation and Conformal Invariance

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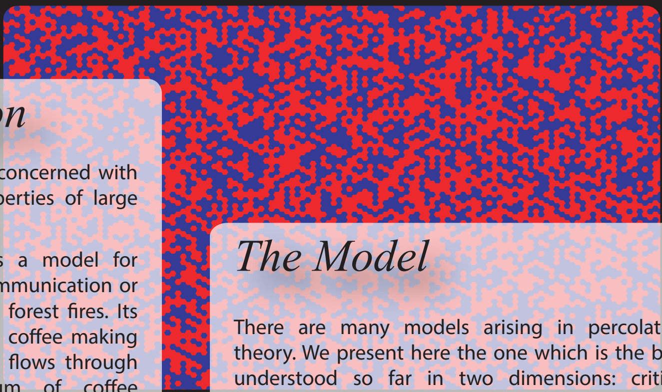


Introduction

Percolation theory is concerned with the connectivity properties of large random graphs.

It is widely used as a model for disordered media, communication or electrical networks or forest fires. Its name stems from the coffee making process, where water flows through the random medium of coffee powder.

From the mathematical point of view it has proved to be a very rich and interesting theory.

Spectacular progress has been made recently towards the understanding of this phenomenon in two dimensions thanks to the use of complex analysis.

Ideas of the proof

The proofs for both the crossing probability and the expected number of clusters share the same structure. The first idea is to let one of the boundary points move inside the domain, making our quantity a function h of the point. Then there are mainly three steps in the proof:

1. One shows Hölder regularity of h, uniformly with respect to the scale. This can be done using correlation inequalities and uniform bounds on certain probabilities and allows to consider convergent subsequences.

2. Boundary conditions are established, by probabilistic and combinatorial means.

3. We show that h is a harmonic function, by finding its harmonic conjugate. This is done by proving a discrete version of Cauchy-Riemann equations:

$$\partial_x h = \partial_y g, \quad \partial_y h = -\partial_x g.$$

We use a discrete Stokes-like formula to transform this discrete equations into a Morera condition, proving analyticity:

$$\frac{1}{2i}\oint_{\partial\Omega} h + ig = \int_{\Omega} \partial(h + ig) = 0.$$

We then show that some Dirichlet or Neumann-Dirichlet problem is solved by our function. Eventually one uses uniqueness results for conformal mappings to conclude about the conformal invariance of our quantity and identify the limit.

The Model

There are many models arising in percolation theory. We present here the one which is the best understood so far in two dimensions: critical percolation on the honeycomb lattice.

Consider the regular honeycomb lattice (the tiling of the plane by regular hexagons) and perform critical percolation on it: color each face either in blue or in red by tossing a fair coin (see the figure above).

We are interested in describing the geometry of the clusters (connected components of a certain color) of such a random configuration.

At the scaling limit (when the size of the hexagons tends to 0), it turns out that these clusters become random fractals, of Hausdorff dimension 91/48 (see below).

The full scaling limit, SLE(6), applications

Thanks to the conformal invariance property, many deep results can be derived about the scaling limit.

In particular the boundary of a cluster can be expressed analytically. Namely it has been shown that random curves satisfying a so-called conformal Markov property belong to a one-parameter family of processes called SLE(κ).

These processes consist in a time-growing collection of compact sets which are the domains of non-definition in the upper half-plane of the solution of the complex stochastic differential equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z$$

where B_t is a standard one-dimensional Brownian motion and κ is a parameter, which is equal to 6 in the percolation case. The yellow fractal on the left is an SLE(6) realisation.

If one looks at the boundaries of the clusters, they appear in the limit as an infinite random collection of nested closed curves which are all described in terms of SLE(6), called loop soup.

This allows to use Itô's calculus to compute quantities related to percolation (for instance the 91/48 dimension of the clusters arises from the main eigenvalue of a certain differential operator).

With these techniques one can also understand near-critical percolation (when one uses almost fair coins for coloring), see [3].

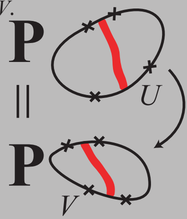
Conformal invariance

The scaling limit of percolation exhibits a remarkable property of conformal invariance: it is invariant under conformal mappings (mappings that locally preserve angles).

As by Riemann's mapping theorem the two-dimensional conformal mappings form an infinite-dimensional family, conformal invariance is a very strong property.

More precisely, consider percolation restricted to a "rectangle" U (a complex domain with four marked points on the boundary) and consider a conformal mapping that maps U onto another rectangle V.

In the limit the probability that a red cluster crosses U and joins two given sides of the rectangle is the same as in the domain V (see[2])!



The expected number of clusters crossing between two sides of a rectangle is also conformally invariant [1] as well as several other quantities.

Moreover exact formulas can be computed for them: if the domain is a disc and c denotes the cross ratio of the four points, the expected number of clusters is given by $-\log(1-c)$.

On the disc the probability that two sides are joined by a red cluster is given by a hypergeometric function of the cross-ratio.

Universality

A very exciting property about these results is that they are conjectured to be universal: the scaling limit of critical percolation on any planar regular lattice should be the same, independently of the particular shape of the lattice. Moreover this is believed to be true on random lattices (e.g. Voronoï tiling of Poisson processes) as well.

Other lattice models (e.g. Ising, Potts, O(n), FK, self-avoiding polymers) are also expected (or shown) to be conformally invariant at critical temperature, leading to SLE with different parameter values.

[1] C. Hongler, S. Smirnov (2008), in preparation.

[2] S. Smirnov (2001), Critical percolation on the plane: Conformal invariance, Cardy's formula, scaling limits, C.R. Acad. Sci. Paris 333, pp. 239-244.

[3] S. Smirnov, W. Werner (2001), Critical exponents for two-dimensional percolation, Mathematical Research Letters, 8, pp. 729-744.

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