LATTICE MODEL EXAMS, JAN 2018

You can re-use without reproving them, results seen in class (unless it is written 'course question'). Do not get stuck on any problem: you certainly do not need to do everything to have the maximal grade. Justify what you write. Total 105 pt.

(1) [14 pt] Consider a simple random walk $\left(S_n^k\right)_{n\geq 0}$ on $\mathbb{Z}$ starting at $k \in \mathbb{Z}$ that has probability p of jumping to the right and probability $q = 1-p$ of jumping to the left ($\mathbb{P}\left\{S_{n+1}^k = S_n^k + 1\right\} = p$ and $\mathbb{P}\left\{S_{n+1}^k = S_n^k - 1\right\} = q$), with $p, q \neq \frac{1}{2}$.

(a) [4 pt] (Course question) Show that the expected number of times the walk visits k is given by $\sum_{n=0}^{\infty} \mathbb{P}\left\{S_n^k = k\right\}$.

(b) [5 pt] Express the expected number of times the walk visits k as the integral of an explicit function, using Fourier analysis technique (as seen in class).

(c) [5 pt] Prove that for $a, b \in \mathbb{Z}$ with $a < b$, the probability that the walk starting from $a \leq k \leq b$ hits b before a is given by

$$\frac{\left(\frac{q}{p}\right)^{k-a} - 1}{\left(\frac{q}{p}\right)^{b-a} - 1}$$

(hint: uniqueness).

(2) [8 pt] Let $G = (V, E)$ be a connected graph and let $a, x, y \in V$ be vertices. Let L_x denote the trajectory of a loop-erased random walk in G (LERW in G) from x to a, and let M_{xy} denote the trajectory of an LERW in G from y to a vertex visited by L_x. Let L_y denote the trajectory of an LERW in G from y to a, and let M_{yx} denote the trajectory of an LERW in G from x to a vertex visited by L_y.

- Using the construction given by the theorem saying that Wilson's algorithm works as a way to sample a Uniform Spanning Tree, show that that the law of $L_x \cup M_{xy}$ and of $L_y \cup M_{yx}$ are the same.

(3) [28 pt] Consider the critical percolation on the hexagonal lattice $\mathbb{C}_\delta$ of mesh size δ, where we color the faces in black or white with probabilities $\frac{1}{2}, \frac{1}{2}$. (Reminder of class notation: for a domain Ω, we denote by Ω_δ a discretization of Ω, such as the largest connected component made of hexagonal faces of $\Omega \cap \mathbb{C}_\delta$)

(a) [7 pt] Consider $\triangleleft \subset \mathbb{C}$ the equilateral triangle of vertices $0, e^{\pi i/6}, e^{-\pi i/6}$, and let D be the image of $\triangleleft$ by the map $z \mapsto z^3$. Consider critical percolation on Ω_δ and compute the limit, as $\delta \to 0$, of the probability that there is a path of black hexagons in D_δ that links the counterclockwise arc $[i, 0]$ to the counterclockwise arc $\left[-\frac{i}{2}, -i\right]$.

(b) [7 pt] (Course question) Let Ω be a Jordan domain (i.e. $\partial\Omega$ is a simple closed curve) and three points $a_1, a_2, a_3 \in \partial\Omega$ in counterclockwise order. For $j = 1, 2, 3$, let $H_j^\delta(z) = \mathbb{P}(Q_j(z))$ where $Q_j^\delta(z)$ is the event that a_j and z are separated (in Ω_δ) from the two points a_{j+1}, a_{j+2} (indices modulo 3) by a black path. State and give a sketch of the proof of the combinatorial identity (often called discrete Cauchy-Riemann equations) which is used to show that the limit of the function $\Phi_\delta(z) = H_1^\delta(z) + e^{2\pi i/3} H_2^\delta(z) + e^{4\pi i/3} H_3^\delta(z)$ as $\delta \to 0$ is holomorphic.

(c) [14 pt] For $n \in \mathbb{N} \setminus \{0\}$, let R_n denote the rectangle $[0, n] \times [0, 1]$ and let $\delta > 0$ be a small fixed mesh size (say $\delta = 0.001$).

(i) [7 pt] Show that the probability P_n that the vertical segments $\{0\} \times [0, 1]$ and $\{n\} \times [0, 1]$ are connected by a black path that stays inside R_n decays exponentially in n, i.e. that there exists $C > 0$ such that $P_n \leq \exp(-Cn)$.

(ii) [7 pt] Show that the probability Q_n that that the vertical segments $\{0\} \times [0,1]$ and $\{n\} \times [0,1]$ are connected by a black path in $\mathbb{C}_\delta$ (i.e. that is not forced to stay in R_n) only decays polynomially, i.e. $Q_n \geq \frac{c}{n^\alpha}$ for some $c, \alpha > 0$.

(4) [30 pt] The Ising model with external magnetic field $h > 0$ consists of ± 1 on a connected graph $G = (V, E)$ consists spins $\sigma_x \in \{\pm 1\}$ for $x \in V$, where the probability of a spin configuration is equal to

$$\mathbb{P}_{\beta,h}\{\sigma\} = \frac{\exp\left(\beta \sum_{\langle i,j\rangle \in E} \sigma_i \sigma_j + h \sum_{i\in V} \sigma_i\right)}{\sum_{\tilde{\sigma}\in\{\pm1\}^V} \exp\left(\beta \sum_{\langle i,j\rangle \in E} \tilde{\sigma}_i \tilde{\sigma}_j + h \sum_{i\in V} \tilde{\sigma}_i\right)}$$

(a) [6 pt] Construct a Markov chain with local updates whose stationary measure is the measure $\mathbb{P}_{\beta,h}$.

(b) [5 pt] For a vertex $x \in V$, compute (and justify) the limit of $\mathbb{E}_{\beta,h}[\sigma_x]$ if we let $\beta \to +\infty$ and then $h \to +\infty$, i.e.

$$\lim_{h\to+\infty} \lim_{\beta\to+\infty} \mathbb{E}_{\beta,h}[\sigma_x].$$

(c) [8pt] For disjoint vertices $x, y \in V$, what are the limits of $\mathbb{E}_{\beta,h}[\sigma_x]$ and of $\mathbb{E}_{\beta,h}[\sigma_x \sigma_y]$ as $\beta \to 0$ and $h > 0$ is fixed?

(d) [11 pt] (A bit harder) On the graph with vertices $\{-3, -2, -1, 0, 1, 2, 3\}$ and edges between nearest neighbors ($v \sim w$ if and only if $|v - w| = 1$), compute the limit

$$\lim_{h\to+\infty} \lim_{\beta\to-\infty} \mathbb{E}_{\beta,h}[\sigma_0].$$

(5) [14 pt] Let $\Omega \subset \mathbb{C}$ be a Jordan domain containing the origin and let Ω_δ be a discretization of Ω by square grid $\delta\mathbb{Z}^2$ of mesh size δ, with boundary $\partial\Omega_\delta$.
Let $\mathcal{B}(\Omega_\delta)$ denote the set of edge collections $\omega \subset \text{Edges}(\Omega_\delta)$ such each vertex of $\Omega_\delta \setminus \partial\Omega_\delta$ belongs to (i.e. is the endpoint of) an even number of edges of ω (there is no constraint for the vertices of $\partial\Omega_\delta$).
Let $\mathcal{B}_0(\Omega_\delta)$ denote the set of edge collections $\tilde{\omega} \subset \text{Edges}(\Omega_\delta)$ such that each vertex of $\Omega_\delta \setminus (\partial\Omega_\delta \cup \{0\})$ belongs to an even number of edges of $\tilde{\omega}$, such that 0 belongs to an odd number of edges of $\tilde{\omega}$ (and there is no constraint for the vertices of $\partial\Omega_\delta$).

(a) [7 pt] Explain why

$$\frac{\sum_{\tilde{\omega}\in\mathcal{B}_0(\Omega_\delta)} (\tanh\beta)^{|\tilde{\omega}|}}{\sum_{\omega\in\mathcal{B}(\Omega_\delta)} (\tanh\beta)^{|\omega|}} \leq 1$$

(b) [7 pt] Show, using results from the course, that for any $\alpha \in]0,1[$, there exists β so that we have

$$\liminf_{\delta\to0} \frac{\sum_{\tilde{\omega}\in\mathcal{B}_0(\Omega_\delta)} (\tanh\beta)^{|\tilde{\omega}|}}{\sum_{\omega\in\mathcal{B}(\Omega_\delta)} (\tanh\beta)^{|\omega|}} \geq \alpha$$

(6) [9pt] (Course question). Let G be a graph with n black vertices $\{b_1, \ldots, b_n\}$ and n white vertices $\{w_1, \ldots, w_n\}$ such that black vertices are only adjacent to white vertices and vice versa . Let $A = (a_{ij})$ be the $n \times n$ matrix with entries given by

$$a_{ij} = \begin{cases} 1 & \text{if } b_i \text{ is adjacent to } w_j \\ 0 & \text{otherwise} \end{cases}$$

Show that the permanent of A is equal to the number of dimer tilings of G.