LATTICE MODELS (MA1, EPFL)

1. Introduction

1.1. What this Class is About. The goal of this class is to give an overview of results about lattice models, which should help you understand their structure, what are the kind of questions to ask, and what are the kind of answers to find. This class is a math class: we prove theorems. However it is a bit different to most undergraduate math classes in the sense that we don't emphasize on definitions, and that we don't try to construct a theory step by step. Rather: we look at fairly concrete questions, study them, and solve them and only later try to infer what is the general philosophy. The general picture is too deep to be studied axiomatically at this point. One of the goals is to teach what are the key mathematical mechanisms to understand lattice models: how should one analyze them, what should one look for, what techniques work.

1.2. Lattice Models. Lattice Models are at the heart of many fields of science:

- Statistical Mechanics: how macroscopic interactions emerges from microscopic physical model
- Quantum Field Theory: high-energy physics (e.g. Higgs boson)
- Biology: models of cells, etc.
- Ecology: models of growing plants
- Economics: models of interacting agents
- Image procesing: probabilistic models of images
- Machine learning: models of computation

1.3. Five Topics. There are mainly five topics that we will discuss in this class, each associated with one or two non-trivial results:

- Simple Random Walk: Recurrence and Transience and PDEs
- Loop-Erased Random Walk and Uniform Spanning Tree: Wilson's Theorem and the Matrix-Tree Theorem
- Percolation: Phase Transition and Cardy's Formula
- Ising Model: Graphical Representations, Sampling and Phase Transition
- Dimer Model: counting dominos

2. Simple Random Walk: Recurrence and Transience

2.1. What is a Simple Random Walk? We look here at discrete-time simple random walks $(X_n)_{n\geq 0}$ on a locally finite graph G, that jump at integer times.

- Simple random walk on $\mathbb{Z}$: we start at 0 and at each time, we jump either left or right, with probabilities $\frac{1}{2}$, independently of the past.
- Simple random walk on $\mathbb{Z}^2$: we start at 0, and at each time, we jump on one of the four neighbors, with probabilities $\frac{1}{4}$, independently of the past.
- On a locally finite graph G: start at an origin vertex, and if $X_n = v$, then X_{n+1} is one of the neighbors of v, chosen with equal probability, independently of the past.

2.2. What are Basic Interesting Questions?

- How does the SRW look on the long term?
- How does a SRW trajectory look from far away?
- Does a SRW ever come back to the origin?
- Are there connections with other random objects?

2.3. **A Recent Result.** At this point, it is worth mentioning a beautiful result of Lawler, Schramm and Werner (we won't prove it, but it is nice to know):

- Consider two independent SRW $(X_n)_n$ and $\left(\tilde{X}_n\right)_n$ on $\mathbb{Z}^2$ from $(0,0)$.
- Call S_n and $\tilde{S}_n$ the set of points visited by X_n and $\tilde{X}_n$ during the times $k = 1, \ldots, n$ (not including $k = 0$).
- We have that $P_n := \mathbb{P}\left\{S_n \cap \tilde{S}_n = \emptyset\right\}$ decays like $n^{-5/8}$ as $n \to \infty$ (i.e. $\log P_n / \log\left(n^{-5/8}\right) \to 1$).

2.4. **Recurrence and Transience: a Theorem attributed to Polya.**

- If we consider the simple random walk on $\mathbb{Z}^d$ starting at the origin. If $d = 1, 2$ with probability 1 *the random walk will come back to the origin, whereas for* $d \geq 3$ *with probability* > 0 *it will not come back to the origin.*
- This result is fairly robust (it works on various graphs); in this class we will only look at the square lattice case.
- The strategy of the proof is the following: reduce the question to one about the expected number of visits of the origin, which is a Markov chain quantity and compute this quantity using Fourier analysis.

2.5. **The Number of Visits to the Origin.**

- Let N_d be the number of times that a SRW on $\mathbb{Z}^d$ comes back to the origin, and let π_d be the probability of ever coming back, i.e. $\pi_d = \mathbb{P}\{N_d \geq 1\}$.
- We have the following facts: if $\pi_d = 1$ then $N_d = \infty$ with probability 1 and if $\pi_d < 1$ then $\mathbb{E}[N_d] < \infty$ (and of course $N_d < \infty$ with probability 1).
- Why? If $\pi_d = 1$, it is obvious: use the Markov property after the first time back to the origin, i.e. the SRW after that time has the same law as a 'new' SRW, which will then again come back to the origin.
- The Markov property also leads to $\mathbb{P}\{N_d \geq k\} = (\pi_d)^k$ hence if $\pi_d < 1$ we get $\sum_{k=1}^{\infty} \mathbb{P}\{N_d \geq k\} = \frac{\pi_d}{1-\pi_d}$.
- The following elementary lemma allows to conclude that $\mathbb{E}[N_d] = \frac{\pi_d}{1-\pi_d} < \infty$ if $\pi_d < 1$.
- Lemma: If $X \geq 0$ is a random integer then $\mathbb{E}[X] = \sum_{j=1}^{\infty} \mathbb{P}\{X \geq j\}$ (works also if both sides are $+\infty$).
- Proof of the lemma:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\mathbb{P}\{X = k\} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{1}_{\{j \leq k\}} \mathbb{P}\{X = k\} = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \mathbb{P}\{X = k\} = \sum_{j=1}^{\infty} P\{X \geq j\}.$$

- As a consequence: proving Polya's Theorem amounts to show that $\mathbb{E}[N_d] = \infty$ if $d \leq 2$ and $\mathbb{E}[N_d] < \infty$ if $d \geq 3$.
- Since $\mathbb{E}[N_d]$ is decreasing with d (think of the projection on the first 3 coordinates), we just need to prove the result in dimensions 2 and 3.

2.6. **Markov Chain.**

- For $x, y \in \mathbb{Z}^d$ and $k \geq 0$, let $Q_k(x, y)$ be the probability that a SRW starting from x arrives at y after exactly k steps.
- We have the 'matrix-multiplication-like' formula: $Q_{k+1}(x, y) = \sum_{z \in \mathbb{Z}^d} Q_k(x, z) Q_1(z, y)$.
- Indeed: we need to jump from x to some point z in k steps and then from z to y in one step, and both probabilities are independent by the Markov property.
- Let us write $P_k(x)$ for $Q_k(0, x)$. By translation invariance we have $Q_k(x, y) = Q_k(0, y - x)$ and hence $P_{k+1}(x) = \sum P_k(z) P_1(x - z)$.
- Let us write P for P_1. By definition $P(x) = \frac{1}{2d}$ if x is one of the $2d$ neighbors of the origin (i.e. $(\pm 1, 0, \ldots, 0)$, $(0, \pm 1, \ldots, 0)$, ... , $(0, \ldots 0, \pm 1)$) and $P(x) = 0$ otherwise.
- If for two functions $f, g : \mathbb{Z}^d \to \mathbb{C}$, we denote by $f \star g$ the ('convolution') function defined by $(f \star g)(x) = \sum_z f(z) g(x - z)$, then $P_k = P^{\star k}$, where $P^{\star k} := P \star \cdots \star P$ (k times).
- Since we have $\mathbb{E}[N_d] = \sum_{k=1}^{\infty} P_k(0)$, what we need to compute is $\sum_{k=0}^{\infty} P^{\star k}(0)$.
- We would prefer a sum of products (geometric series) to a sum of convolutions. How to transform convolutions into products? Fourier analysis.

2.7. **Fourier Analysis in $d = 1$.**

- A function $f : \mathbb{Z} \to \mathbb{C}$ is nothing but a (bi-infinite) series. Let us assume that everything converges. [this is the case in our case, the support of f is finite]
- If we consider a bi-infinite series f, we can form a corresponding Fourier series $\mathcal{F}f(x) := \sum_k f(k) e^{ikx}$, which is a 2π-periodic function.
- We have the classical inversion formula: from $\mathcal{F}f(x)$, we can recover $f(k)$ by $f(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}f(x) e^{-ikx}\mathrm{d}x$ (why?).
- If $h = f \star g$, for $f, g : \mathbb{Z} \to \mathbb{C}$, we have that $\mathcal{F}(h) = \mathcal{F}(f)\mathcal{F}(g)$.
- Proof: take $\sum_k f(k) e^{ikx} \sum_\ell g(\ell) e^{i\ell x}$, make a change of variable $\sum_k f(k) e^{ikx} \sum_\ell g(\ell - k) e^{i(\ell-k)x}$ and re-arrange into $\sum_k \left(\sum_\ell f(k) g(\ell - k)\right) e^{ikx}$, which gives $\mathcal{F}h(x)$.
- So for our problem (assuming things converge) we have $\mathcal{F}\left(P^{\star k}\right)(x) = \left(\mathcal{F}(P)\right)^k(x)$ and hence
$$\mathcal{F}\left(\sum_{k=0}^{\infty} P^{\star k}\right)(x) = \frac{1}{1-\mathcal{F}(P)}(x)$$
- Since $\mathcal{F}P(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos(x)$, we have $\mathcal{F}\left(\sum_{k=0}^{\infty} P^{\star k}\right)(x) = \frac{1}{1-\cos(x)}$.
- So, if we want to compute $\mathbb{E}\left[N_{d=1}\right] = \sum_{k=0}^{\infty} P^{\star k}(0)$, we use the inversion formula and obtain $\int_{-\pi}^{\pi} \frac{1}{1-\cos(x)}\mathrm{d}x$.
- Since $\cos(x) = 1 - \frac{1}{2}x^2 + \cdots$ near $x = 0$, this integral is divergent, therefore the expectation is infinite.
- How to make this more rigorous: dampen the sum by adding a mass term $\sum_{k=0}^{\infty} \lambda^k P^{\star k}(0)$ for $\lambda < 1$, get $\int_{-\pi}^{\pi} \frac{1}{1-\lambda\cos(x)}\mathrm{d}x$, which is finite (and absolutely convergent) and then let $\lambda \to 1$ and use monotone convergence.

2.8. **Fourier Analysis in $d \geq 2$.**

- The analysis is pretty much the same.
- A function $f : \mathbb{Z}^d \to \mathbb{C}$ gives rise to a Fourier series $\mathcal{F}f(\mathbf{x})$ of d variables defined by $\sum_{\mathbf{k}\in\mathbb{Z}^d} f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$ (we wrote $\mathbf{k}\cdot\mathbf{x} := k_1x_1 + \cdots + k_dx_d$), and this function is periodic in each variable.
- The inversion formula is given by $f(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \mathcal{F}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\mathrm{d}x_1\cdots\mathrm{d}x_d$ (same proof as before).
- For exactly the same reasons as before $\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)$.
- Hence for our problem, we have as before $\mathcal{F}\left(\sum_{k=0}^{\infty} P^{\star k}\right)(\mathbf{x}) = \frac{1}{1-\mathcal{F}(P)}(\mathbf{x})$.
- Now $\mathcal{F}(P)(\mathbf{x}) = \frac{1}{d}\sum_{j=1}^{d}\cos(x_j)$ and hence we get (formally)
$$\mathbb{E}\left[N_d\right] = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \frac{1}{1-\frac{1}{d}\sum\cos(x_j)}\mathrm{d}x_1\cdots\mathrm{d}x_d$$
- Is this integral divergence or convergent? Obviously, the only singularity in the hypercube $[-\pi,\pi]^d$ is at $x_1 = \cdots = x_d = 0$, so let's see what happens there.
- Near $(0,\ldots,0)$, we have $\frac{1}{d}\sum\cos(x_j) = 1 - \frac{1}{2d}\sum_{j=1}^{d} x_j^2 + \cdots$ and hence we should study the d-dimensional integral of $\frac{1}{x^2}$ around the origin.
- In $d = 2$, we can do a polar change of variable, to get $\int_0^\epsilon\int_0^{2\pi}\frac{1}{r^2}r\mathrm{d}r\mathrm{d}\theta$, which is infinite because $\int_0^\epsilon \frac{1}{r}\mathrm{d}r = \infty$, but 'barely infinite'.
- In $d = 3$, we can do a spherical change of variable to get $\int_0^\epsilon\int_0^{2\theta}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{r^2}r^2\cos\varphi\mathrm{d}\varphi\mathrm{d}\theta\mathrm{d}r$, which is obviously finite.
- To make things rigorous, we can use the exact same trick as in 1D.

2.9. **Expected time to come back.** In $d = 1$, the time to come back to 0, T, is finite almost surely, but
$$\mathbb{E}[T] = \infty.$$
Two proofs: a combinatorial proof, one using discrete PDEs

2.9.1. *Combinatorial proof.* Consider a simple random walk $(S_n)_n$ starting from 0.

- $\mathbb{E}[T] = \sum_{k\geq 0}\mathbb{P}[T > k]$: we need to study how $\mathbb{P}[T > k]$ decays.
- First step of the RW: $\mathbb{P}[T > k] = \frac{1}{2}\mathbb{P}[T > k \mid S_1 = 1] + \frac{1}{2}\mathbb{P}[T > k \mid S_1 = -1]$
- Symmetry of the RW wrt 0: $\mathbb{P}[T > k \mid S_1 = 1] = \mathbb{P}[T > k \mid S_1 = -1]$
- Invariance by shifting the time: $\mathbb{P}[T > k \mid S_1 = 1] = \mathbb{P}_1[T > k - 1]$ where $\mathbb{P}_1$ means that we consider the random walk starting from 1.
- Hence we need to study how $\mathbb{P}[T > k \mid S_0 = 1]$ decays as $k \to \infty$.

- We compute $\mathbb{P}_1[T \le k] := \mathbb{P}[T \le k \mid S_0 = 1]$:
  - $\mathbb{P}_1[T \le k] = \sum_{x\in\mathbb{Z}} \mathbb{P}_1[T \le k \;\&\; S_k = x]$
  - If $x \le 0$: $\mathbb{P}_1[T \le k \;\&\; S_k = x] = \mathbb{P}_1[S_k = x]$ since the RW has to cross the horizontal line $y = 0$. We can use the translation invariance: $\mathbb{P}_1[T \le k \;\&\; S_k = x] = \mathbb{P}_0[S_k = x - 1]$.
  - If $x > 0$: we have:
    * By definition $\mathbb{P}_1[T \le k \;\&\; S_k = x] = \frac{1}{2^k}\#\{\text{walks of length } k \text{ from } 1 \text{ to } x \text{ which cross } y = 0\}$
    * If we have a walk $x_0 = 1, \dots, x_k = x$ which crosses $y = 0$, we can consider the first time it crosses $y = 0$, say n, and we can associate the walk $\tilde{x}_0 = -x_0, \dots, \tilde{x}_n = -x_n, x_{n+1}, \dots, x_k$. This gives a bijection [*the reflection principle*]:

$$\{\text{walks of length } k \text{ from } 1 \text{ to } x \text{ which cross } y = 0\} \mapsto \{\text{walks of length } k \text{ from } -1 \text{ to } x\ \}.$$

    * Hence

$$\mathbb{P}_1[T \le k \;\&\; S_k = x] \quad = \quad \frac{1}{2^k}\#\{\text{walks of length } k \text{ from } -1 \text{ to } x\ \} = \mathbb{P}_{-1}[S_k = x] = \mathbb{P}_0[S_k = x + 1].$$

  - $\mathbb{P}_1[T \le k] = \sum_{x\in\mathbb{Z}} \mathbb{P}_1[T \le k \;\&\; S_k = x]$ is equal to

$$\sum_{x\le 0} \mathbb{P}_1[T \le k \;\&\; S_k = x] + \sum_{x>0} \mathbb{P}_1[T \le k \;\&\; S_k = x]$$

  and thus, equal to:

$$\sum_{x\le 0} \mathbb{P}_0[S_k = x - 1] + \sum_{x>0} \mathbb{P}_0[S_k = x + 1] = 1 - \mathbb{P}_0[S_k = 0] - \mathbb{P}_0[S_k = 1].$$

- Note that $\mathbb{P}_0[S_{2k} = 1] = 0$ since the random walk can not come back to 1 in an even number of steps.
- Hence $\mathbb{P}_1[T > 2k] = \mathbb{P}_0[S_{2k} = 0] \sim C/\sqrt{k}$ [Exercise sheet].
- $\sum_{k\ge 1} \frac{1}{\sqrt{k}} = \infty$, so $\mathbb{E}[T] \ge \sum_{2k\ge 0} \mathbb{P}[T > 2k + 1] = \sum_{2k\ge 0} \mathbb{P}_1[T > 2k] = \infty$.

2.9.2. *Proof using PDEs:*

- We consider $A = \{1, \dots, N - 1\}$ and $\partial A = \{0, N\}$, we compute $\mathbb{E}_x[T_{0,N}]$ where $x \in A \cup \partial A$ and $T_{0,N}$ is the first time $n \ge 0$ at which the simple random walk exits from A.
- The function $f : x \mapsto \mathbb{E}_x[T_{0,N}]$ satisfies a simple discrete PDE, with following bulk and boundary conditions.
- Boundary conditions: $f(0) = f(N) = 0$.
- Bulk conditions: obtained by looking at the first step of the random walk: $f(x) = \frac{1}{2}f(x+1)+\frac{1}{2}f(x-1)+1$ for $x \in A$, which can be written as $\Delta f(x) = -1$ where

$$\Delta f(x) = \frac{1}{2}f(x + 1) + \frac{1}{2}f(x - 1) - f(x).$$

- Δ is a discretization of the Laplacian [see next section]: in the continuous, we look for a function which has a constant second derivative, and is 0 at 0 and at N. It must be a polynomial of degree 2 which vanishes at 0 and N: $g(x) = x(N - x)$ would be a candidate. Indeed, we can check that g satisties the bulk and boundary conditions for $x \in A$ and $x \in \partial A$.
- Unicity of the discrete PDE: there is a unique solution to

$$\begin{cases} f(0) = f(N) = 0, & \\ \Delta f(x) = -1 & \text{for } x \in A, \end{cases}$$

  If two functions satisfy this discrete PDE, their difference ψ is harmonic (*i.e.* $\Delta\psi = 0$) and is equal to 0 at 0 and N. Its maximal value is attained at x_0, but since $\Delta\psi(x_0) = 0$, it must be also attained on the neighbours of x_0, and so on, until we reach 0 or N: the maximal value of ψ is 0. Using the same argument for $-\psi$, we get that the minimal value of ψ is 0. Hence $\psi = 0$ and the two solutions are equal.
- Hence $\mathbb{E}_x[T_{0,N}] = x(N - x)$.
- Since the time T to return to 0 for the RW starting at 1 satisfies $\mathbb{E}_1[T] > \mathbb{E}_1[T_{0,N}] = N-1$, by considering the limit $N \to \infty$ we get $\mathbb{E}_1[T] = \infty$.
- From this, using the usual argument [first step and symmetry wrt to $y = 0$] we get that the return time to 0 for the RW starting at 0 has an infinite expectation.

2.10. **An other proof of the recurrence for $d = 1$.**

- We can use the same idea than in 2.9.2. to prove the fact that $T < \infty$ in dimension 1.
- We consider $A = \{1, \dots, N-1\}$ and $\partial A = \{0, N\}$, and compute $f(x) = \mathbb{P}_x[S_{T_{0,N}} = 0]$.
- The function f satisfies a simple discrete PDE, with bulk and boundary conditions.
- Boundary conditions: $f(0) = 1$, $f(N) = 0$
- Bulk conditions: with the same arguments $\Delta f(x) = 0$
- We think about the continuous version of the PDE: we look for a function whose second derivative vanishes. It is an affine function and because of the boundary conditions, it should be $\frac{N-x}{N}$. We can verify easily that $g(x) = \frac{N-x}{N}$ is a solution of the discrete PDE.
- Same argument as before: the solution of
$$\begin{cases} f(0) = 1 \\ f(N) = 0 \\ \Delta f(x) = 0 \quad \text{for } x \in A \end{cases}$$
is unique.
- So $\mathbb{P}_x[S_{T_{0,N}} = 0] = \frac{N-x}{N}$.
- In particular $\mathbb{P}_1\left[T < \infty\right] \geq \mathbb{P}_1[S_{T_{0,N}} = 0] = \frac{N-1}{N}$ which converges to 1 as $N \to \infty$.
- From this, using the usual argument [first step and symmetry wrt to $y = 0$], we get that the return time to 0 for the RW starting at 0 is almost surely finite.

1. Simple Random Walks and PDEs

1.1. Goal of this lecture.

- Generalize the connection we have seen between simple random walks and some Discrete Partial Differential Equations, mainly the Laplace equation $\Delta f = 0$, the Poisson equation $\Delta f = \rho$ and the heat equation $\partial_t f = \Delta f$.
- To explain why the discrete solutions are approximations of the continuous equations.
- This connection is useful to
  - study the continuous limit of random walks (in particular, calculate things on it),
  - understand Laplace equation and the heat equation (in particular, obtain qualitative results easily).

1.2. The discrete Laplacian.

- Consider a finite connected graph $\mathbb{G} \subset \mathbb{Z}^d$ with boundary $\partial\mathbb{G}$ (the points which are not in $\mathbb{G}$ but which are the neighbours of a point in $\mathbb{G}$).
- Given a function $f : \mathbb{G} \cup \partial\mathbb{G} \to \mathbb{R}$, the Laplacian of f at x is defined for any $x \in G$ as

$$\Delta f(x) = \frac{1}{2d} \sum_{y \sim x} \left[f(y) - f(x) \right].$$

- This defines an operator $\Delta : \mathbb{R}^{\mathbb{G}\cup\partial\mathbb{G}} \to \mathbb{R}^{\mathbb{G}}$ which is not injective (by comparing the dimensions). In particular, if you consider the constant functions, they are all satisfying $\Delta f = 0$.
- If we have boundary conditions, i.e. $F \in \mathbb{R}^{\partial\mathbb{G}}$, we can define the Laplacian with boundary F: it is an operator on the space of functions $f : \mathbb{G} \to \mathbb{R}$ defined as

$$\Delta^F f(x) = \frac{1}{2d} \sum_{y \sim x} \left[f(y) - f(x) \right]$$

  where we take the convention that $f(y) = F(y)$ if $y \in \partial\mathbb{G}$.
- This defined an operator $\Delta^F : \mathbb{R}^{\mathbb{G}} \to \mathbb{R}^{\mathbb{G}}$. We can also see it naturally as an operator on the functions on $\mathbb{G} \cup \partial\mathbb{G}$ which are equal to F on the boundary.
- Proposition: the operator Δ^F is a bijection.
- Indeed, since the source and target spaces have the same finite dimension, we only need to prove that it is injective. If $\Delta^F f = \Delta^F g$, then we can see $f - g$ as a function on $\mathbb{G} \cup \partial\mathbb{G}$ which is equal to 0 on the boundary and such that $\Delta(f - g) = 0$. As before, we can use the *maximum principle :* the maximum and minimum of $f - g$ has to be reached on the boundary. [consider x_0 at which the maximum is reached, if it is on the boundary it is fine, if not, we can use the mean value property of the function (i.e. the value at x_0 is the mean of the values at the neighbours) to see that the maximum is reached also on the neighbours, so on and so forth, until we reach the boundary: thus the max is reached on the boundary]. Since the function is 0 on the boundary, $f - g = 0$ on $\mathbb{G}$.
- Actually (exercise), the discrete Laplacian with 0 boundary conditions is negative definite and so is its inverse.
- Thus we can invert the Laplacian with boundary conditions: for any function F on $\partial\mathbb{G}$, any function ρ on $\mathbb{G}$, there exists a unique function f on $\mathbb{G}$ such that

$$\Delta^F f = \rho.$$

1.3. Discrete Partial Differential Equations:

- The equation $\Delta^F f = \rho$ can be written as

$$\begin{cases} \Delta f(x) = \rho & \text{on } \mathbb{G}, \\ f(x) = F(x) & \text{on } \partial\mathbb{G}, \end{cases} \tag{1.1}$$

  where F is a given function on $\partial\mathbb{G}$ and ρ a function on $\mathbb{G}$.
- We want to give a probabilitistic interpretation of the solution f using Random Walks.
- The problem is linear: if we have the solutions f_1 and f_2 for the data (ρ_1, F_1) and (ρ_1, F_2), then $\alpha f_1 + \beta f_2$ is the unique solution for $(\alpha\rho_1 + \beta\rho_2, \alpha F_1 + \beta F_2)$.
- Need to study the solution associated to $(0, F)$ and $(\rho, 0)$: the discrete partial differential equations are respectively the Laplace equation with Dirichlet boundary condition F, and the Poisson equation with source term ρ.

1.3.1. *The Laplace equation with Dirichlet boundary conditions.*

- We consider the Laplace equation with Dirichlet boundary conditions:

$$\begin{cases} \Delta f(x) = 0 & \text{on } \mathbb{G}, \\ f(x) = F(x) & \text{on } \partial\mathbb{G}, \end{cases} \tag{1.2}$$

  where F is a given function on $\partial\mathbb{G}$.
- This kind of problem is particularly useful to extend a function on $\partial\mathbb{G}$ to $\mathbb{G}$ in a natural manner:
  - for instance if $f : \partial\mathbb{G} \to \mathbb{R}$ is the value of an eletric potential on the boundary of a grid $\mathbb{G}$, then F is the value of the potential inside the grid,
  - another (related) example: if we put some temperature on the boundary of a grid and wait until the temperature equilibrates inside, it will be given by F.
- We have seen that the problem is linear.
- We only need to understand the solution of

$$\begin{cases} \Delta f(x) = 0 & \text{on } \mathbb{G}, \\ f(x) = \delta_{x=y} & \text{on } \partial\mathbb{G}, \end{cases} \tag{1.3}$$

  where y is a point of the boundary.
- The solution of Equation (1.3) is denoted by $H(x, y)$. More generaly, the *discrete Harmonic Measure* $H(x, B)$ is the solution of the Laplace Equation with boundary condition $\delta_{x\in B}$ with $B \subset \partial\mathbb{G}$. It is the unique discrete harmonic function, i.e.

$$H(x, B) = \frac{1}{2d} \sum_{y\sim x} H(y, B)$$

  for $x \in \mathbb{G}$ which is equal to 1 on B, 0 on $\partial\mathbb{G} \setminus B$.
- Any solution to the general Laplace equation is of the form

$$f(x) = \sum_{y\in\partial\mathbb{G}} H(x, y)F(y),$$

  in particular $H(x, B) = \sum_{y\in B} H(x, y)$.

1.3.2. *The Poisson equation.*

- We consider the Poisson equation:

$$\begin{cases} \Delta f(x) = \rho(x) & \text{on } \mathbb{G} \\ f(x) = 0 & \text{on } \partial\mathbb{G} \end{cases}, \tag{1.4}$$

  where ρ is a given function on $\mathbb{G}$.
- We have seen that the problem is linear.
- We only need to understand the solution of

$$\begin{cases} \Delta f(x) = -\delta_{x=y} & \text{on } \mathbb{G} \\ f(x) = 0 & \text{on } \partial\mathbb{G} \end{cases}, \tag{1.5}$$

  where y is a point in $\mathbb{G}$.
- The solution of Equation (1.5) is the Green Function, denoted by $G(x, y)$.
- Any solution to the general Poisson equation is of the form

$$f(x) = -\sum_{y\in\partial\mathbb{G}} G(x, y)\rho(y).$$

- Consider the matrix of Δ^0 (indices are the vertices of $\mathbb{G}$, and we consider null boundary conditions), that we denote also by Δ^0, and consider the matrix $G = (G(x, y))_{x,y}$:

$$\Delta^0 G = -\mathrm{Id}$$

  i.e. the matrix $G = -\left(\Delta^0\right)^{-1}$.

1.3.3. *Random Walk Interpretation of the Harmonic Measure and of the Green Function.*

- It remains to give a probabilistic interpretation of $H(x,y)$ and $G(x,y)$.
- We consider a simple RW $X := (X_n)_n$ on $\mathbb{Z}^d$ which starts at $x \in \mathbb{G}$, $T_{\partial\mathbb{G}}$ the first time X visits the boundary.
- $H(x,y) = \mathbb{P}_x\left[X_{T_{\partial\mathbb{G}}} = y\right]$, i.e. the probability that the RW starting at x exits at y. More generally, $H(x,B) = \mathbb{P}_x\left[X_{T_{\partial\mathbb{G}}} \in B\right]$, i.e. the probability that the RW starting at x exits in B.
  - Why ? Using the unicity of the solution of Equation (1.3), we only need to prove that $\mathbb{P}_x\left[X_{T_{\partial\mathbb{G}}} = y\right]$ is solution of Equation (1.3). Considering the first step of the RW $\to$ it is discrete harmonic. The boundary condition is trivial.
- $G(x,y) = \mathbb{E}\left[\sum_{n=0}^{T_{\partial\mathbb{G}}-1} \delta_{X_n=y}\right]$, i.e. the expected number of visits at y of the RW starting at x.
  - Why ? Same argument.
- The general solution of Equation (1.1) is given by

$$f(x) = -\mathbb{E}\left[\sum_{n=0}^{T_{\partial\mathbb{G}}-1} \rho(X_n)\right] + \mathbb{E}\left[F(X_{T_{\partial\mathbb{G}}})\right].$$

1.4. **Discrete Heat Equation.**

1.4.1. *Existence, Unicity and the Probabilistic Interpretation.*

- We consider functions $f : \mathbb{N} \times \mathbb{G} \to \mathbb{R}$, and the discrete time differential $\partial_n f(n,x) = f(n+1,x) - f(x)$.
- We want to study the Discrete Heat Equation.

$$\begin{cases} \partial_n f(n,x) = \Delta f(n,x) & \text{on } \mathbb{N} \times \mathbb{G}, \\ f(0,x) = F(x) & \text{for } x \in \mathbb{G} \cup \partial\mathbb{G}, \\ f(n,x) = G(n,x) & \text{for } (n,x) \in \mathbb{N}^* \times \partial\mathbb{G}. \end{cases}$$

- Solution exists and is unique: $\partial_n f = \Delta f(x)$, i.e. $f(n+1,x) = f(x) + \Delta f(x)$ gives us an algorithm to define the function f at time $n+1$ if we know the function at time n. Since the initial condition is given (i.e. $f(0,\cdot)$) we conclude.
- Probabilistic interpretation is $f(n,x) = \mathbb{E}\left[F(X_n)\, 1_{n \le T_{\partial\mathbb{G}}} + G\left(n - T_{\partial\mathbb{G}}, X_{T_{\partial\mathbb{G}}}\right) 1_{T_{\partial\mathbb{G}} < n}\right]$ (proof in the Exercise Sheet n$^{\text{o}}$4)
- We focus on computing the solution to the Discrete Heat Equation in a special case ($F = \delta_{x=x_0}$ and $G = 0$) in order to compute $\mathbb{P}_{x_0}\left[X_{n \wedge T_{\partial\mathbb{G}}} = y\right]$ ($n \wedge T_{\partial\mathbb{G}}$ is the minimum between the two values), when X is a simple random walk on $\mathbb{G} = \{1, \dots, N-1\}$ with boundary $\partial\mathbb{G} = \{0, N\}$.

1.4.2. *Solving the Discrete Heat Equation; Density of the RW killed at* 0 *and* N*:*

- Consider X a simple random walk on $\mathbb{G} = \{1, \dots, N-1\}$ with boundary $\partial\mathbb{G} = \{0, N\}$, which starts at $x_0 \in \mathbb{G}$.
- Theorem: for any $y \in \mathbb{G}$:

$$\mathbb{P}_{x_0}\left[X_{n \wedge T_{\partial\mathbb{G}}} = y\right] = \frac{2}{N} \sum_{j=1}^{N-1} \Phi_j(x) \left[\cos\left(\frac{j\pi}{N}\right)\right]^n \Phi_j(y),$$

  with $\Phi_j(x) = \sin\left(\frac{\pi j x}{N}\right)$.
- How to prove that:
  - Consider $f(n,y) = \mathbb{P}_{x_0}\left[X_{n \wedge T_{\partial\mathbb{G}}} = y\right]$ for $y \in \mathbb{G}$ and $f(n,y) = 0$ for $y \in \partial\mathbb{G}$.
  - Show that $f(n,y)$ satisfies the discrete Heat Equation.
  - Show that the algorithm consists in multiplying the vector $f(n,\cdot)$ at each step by a fixed symmetric matrix Q, hence $f(n,\cdot) = Q^n f(0,\cdot)$.
  - Diagonalize Q: find the eigenvalues λ_i and the eigenvectors Φ_i.
  - The eigenvectors are orthogonal.
  - Decompose the initial density $\delta_{y=x_0}$ on the eigenvectors: $\delta_{y=x_0} = \sum a_i \Phi_i$. This is done by considering the scalar product (the e.v. are orthogonal)

$$\langle \delta_{y=x_0}, \Phi_j \rangle = \left\langle \sum a_i \Phi_i, \Phi_j \right\rangle = a_j \left\|\Phi_j\right\|^2$$

    and by computing $\langle \delta_{y=x_0}, \Phi_j \rangle$ and $\left\|\Phi_j\right\|^2$.

– We thus obtain:

$$\mathbb{P}_{x_0}\left[X_{n\wedge T_{\partial\mathbb{G}}}=y\right]=Q^n\delta_{y=x_0}(y)=\sum a_i\lambda_i^n\Phi_i(y).$$

- First step: $f(n,y)$ satisfies the Heat Equation: indeed, the probability that X is at y at time $n+1$ is the sum over the neighbours z of y of the probability that X is at z at time n and that it jumped from z to y. We obtain $f(n+1,y)=\frac{1}{2d}\sum_{z\sim y}f(n,z)$ or $\partial_n f(n,y)=\Delta f(n,y)$.
- Second step: the discrete Heat Equation can be written as $f(n+1,y)=\frac{1}{2d}\sum_{z\sim y}f(n,z)$. Hence we see that we multiply $(f(n,z))_{z\in\mathbb{G}}$ by a matrix $Q=\Delta^0+\mathrm{Id}$ in order to get $(f(n+1,z))_{z\in\mathbb{G}}$: by induction

$$f(n,y)=\left(Q^n f(0,\cdot)\right)_y.$$

- Third step: we consider $\Psi_\theta(x)=\sin(\theta x)$, in order to have the 0 boundary condition, $\sin(\theta N)=0$ hence $\theta N=k\pi$ or $\theta_k=\frac{k\pi}{N}$. Define $\Phi_k(x)=\sin(\theta_k x)$, and denote by $\Im$ the imaginary part, then:

$$\Phi_k(x\pm 1)=\Im(e^{i\theta_k(x\pm 1)})=\Im(e^{i\theta_k x}e^{\pm ix})=\sin(\theta_k x)\cos(\theta_k)\pm\cos(\theta_k x)\sin(\theta_k)$$

hence $\Phi_k(x+1)+\Phi_k(x-1)=2\cos(\theta_k)\Phi_k(x)$. For each k, we have found an eigenvector $\Phi_k(x)=\sin(\frac{k\pi}{N}x)$ associated with the eigenvalue $\cos(\frac{k\pi}{N})$.
- Forth step: the space of functions we consider is a $N-1$ dimensional vector space (it is the space of functions on $\{1,\dots,N-1\}$). We should have $N-1$ eigenvectors. In fact $\Phi_{k+N}(\cdot)=-\Phi_k(\cdot)$, hence we can consider only $k\in\{0,\dots,N-1\}$. Besides, $\Phi_0(\cdot)=0$, so we only consider Φ_k where $k\in\{1,\dots,N-1\}$. We have $N-1$ eigenvectors. Their eigenvalues are distincts: since Q is symmetric, they are orthogonal. [Use $\Phi_k^t(Q\Phi_l)=(Q\Phi_k)^t\,\Phi_l$]. So we have a basis of eigenvectors.
- Fifth step: we decompose $\delta_{y=x_0}=\sum a_i\Phi_i$. We only need to compute:
  – $\langle\delta_{y=x_0},\Phi_j\rangle$: This is simply $\Phi_j(x_0)$.
  – $\|\Phi_j\|^2$: We need to compute

$$\sum_{x=1}^{N-1}\Phi_j(x)^2=\sum_{x=1}^{N-1}\sin(\frac{j\pi}{N}x)^2.$$

Using $\sin(\theta)^2=\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^2=-\frac{1}{4}\left[e^{2i\theta}+e^{-2i\theta}-2\right]=\frac{1}{2}-\frac{1}{2}\cos(2\theta)$, we get

$$\sum_{x=1}^{N-1}\Phi_j(x)^2=\frac{N-1}{2}-\frac{1}{2}\sum_{y=1}^{N-1}\cos(\frac{2j\pi}{N}y),$$

and since $\cos(\frac{2j\pi}{N}y)=\Re(e^{i\frac{2j\pi}{N}y})$, we have

$$\frac{1}{2}\sum_{y=1}^{N-1}\cos(\frac{2j\pi}{N}y)=\frac{1}{2}\Re\left(\sum_{y=1}^{N-1}e^{i\frac{2j\pi}{N}y}\right).$$

Besides, $\left(\sum_{y=1}^{N-1}e^{i\frac{2j\pi}{N}y}\right)(1-e^{i\frac{2j\pi}{N}y})=e^{i\frac{2j\pi}{N}y}-1$ hence $\sum_{y=1}^{N-1}e^{i\frac{2j\pi}{N}y}=-1$ and thus

$$\frac{1}{2}\sum_{y=1}^{N-1}\cos(\frac{2j\pi}{N}y)=-\frac{1}{2}.$$

This yields $\sum_{x=1}^{N-1}\Phi_j(x)^2=\frac{N}{2}$, or $\|\Phi_j\|^2=\frac{N}{2}$.
- Hence the decomposition is given by

$$\delta_{y=x_0}=\sum a_i\Phi_i$$

where $a_i=\frac{\langle\delta_{y=x_0},\Phi_j\rangle}{\|\Phi_j\|^2}=\frac{2}{N}\Phi_j(x_0)$.
- The solution is thus

$$\mathbb{P}_{x_0}\left[X_{n\wedge T_{\partial\mathbb{G}}}=y\right]=\left(Q^n\delta_{\cdot=x_0}\right)_y=\frac{2}{N}\sum_{j=1}^{N-1}\Phi_j(x_0)\left(\cos\left(\frac{j\pi}{N}\right)\right)^n\Phi_j(y).$$

2. Convergence

2.1. **Continuous Operators.**

- Let us now consider the grid $\delta\mathbb{Z}^d$, i.e. the lattice rescaled by a factor $\delta > 0$, called the mesh size: the distance between two adjacent vertices becomes δ. What happens as $\delta \to 0$?
- Let us look at a smooth bounded domain $G \subset \mathbb{R}^d$ and at its discretization $G_\delta := G \cap \delta\mathbb{Z}^d$. Whenever needed, identify points of G with the closest vertices of G_δ.
- We have that if $F : G \to \mathbb{R}$ is a $\mathcal{C}^2$ function then $\frac{1}{\delta^2}\Delta_{G_\delta} F(x) \to \frac{1}{2}\Delta F(x)$ for all $x \in G$, where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. Why? Taylor expand F.

2.2. **Convergence of Solutions.**

- There are many questions and many answers of this type. Let us ask the most natural one: consider a family of discrete approximations $(G_\delta)_{\delta>0}$ of a smooth domain of $\mathbb{R}^d$ and fix pieces of the boundary $B_\delta := \partial G_\delta \cap \Lambda$ for an open set $\Lambda \subset \mathbb{R}^d$.
- Statement: as $\delta \to 0$ the harmonic measure $H_{G_\delta}(z, B_\delta)$ converges to $H_G(z, B)$, where $H_G(\cdot, B)$ is continuous harmonic.
- There are many other similar statements, which can be proven with similar techniques.

2.3. **Convergence Strategy.**

- Interpolation: we first interpolate our functions to obtain continuous functions.
- Bounded values: we need to make sure that the functions do not “explode”: since they are $[0, 1]$ valued, that is fine.
- Regularity estimates: we need to make sure that the function is not ’too crazy’, i.e. ’it doesn’t wiggle too much’.
- We can apply Arzelà-Ascoli precompactness results: for any subsequence, we can extract a convergent subsubsequence. If we show that the limit is always the same for any subsubsequence, then it shows that the sequence was infact converging towards this limit.
- Show that the limit of each converging subsequence satisfies enough conditions, so that it has to be unique: show that it satisfies the continuous Laplace equation, with boundary condition 1 on B and 0 on $\partial G \setminus B$; i.e. it is smooth on G, radially continuous everywhere on ∂G except at the endpoints of B, harmonic on G and takes the good boundary values.
  - Harmonicity will come from the fact that the discrete functions are discrete harmonic.
  - The boundary conditions is satisfied since the discrete functions satisfy such discrete boundary conditions.
  - A priori boundary estimates: make sure that as we approach B, the discrete functions get uniformly close to 1, and similarly for $\partial G \setminus B$.
- Uniqueness result: there is a unique continuous harmonic function $H_G(z, B)$ with the relevant boundary conditions.
- Conclusion: the sequence $H_{G_\delta}(z, B_\delta)$ converge to the unique solution of the continuous Laplace equation, i.e. to $H_G(z, B)$.

2.4. **Regularity estimates.**

- The idea is to use a discrete Harnack-type inequality to control the discrete partial derivatives of discrete harmonic functions.
- For a function $f : \delta\mathbb{Z}^d \to \mathbb{R}$, and $k \in \{1, \ldots, d\}$, we denote by $\partial_\delta^k f$ the partial derivative defined by $\partial_\delta^k f(x) = f(x + \delta e_k) - f(x)$, where e_k is the k-th vector of the canonical basis.
- The *Harnack inequality*: for a discrete harmonic function f defined on $B_\delta(x, r) := \left\{z \in \mathbb{Z}^d : |z - x| \leq r\right\}$, there exists $C > 0$ such that for all $k \in \{1 \ldots, d\}$,

$$\left|\partial_\delta^k f(x)\right| \leq C \max_{z \in B_\delta(x,r)} |f(z)|$$

2.5. **Boundary control.**

- We already know our functions are between 0 and 1 and behave nicely inside; what we need to know is that they don’t suddenly jump as we approach the boundary
- First, notice that they *are* going to jump when we go from B_δ to its complementary $C_\delta := \partial G_\delta$
- So, let us just look at $B_\delta^r := \{z \in B_\delta : \mathrm{d}(z, C_\delta) \geq r\}$ and $C_\delta^r := \{z \in C_\delta : \mathrm{d}(z, B_\delta) \geq r\}$ for some small fixed $r > 0$.

- What we want to show (r is fixed): for any $\epsilon > 0$, there exists $\rho > 0$ such that for all $\delta > 0$ and $z \in G_\delta$ with $\mathrm{d}\left(z, B_\delta^r\right) \leq \rho$ then $\left|H_{G_\delta}\left(z, B_\delta\right) - 1\right| \leq \epsilon$ and for all $z \in G_\delta$ with $\mathrm{d}\left(z, C_\delta\right) \leq \rho$ then $\left|H_\delta\left(z, B_\delta\right)\right| \leq \epsilon$.
- Let us prove this in 2D: the inequality is called the discrete Beurling estimate.

2.6. **Discrete Beurling estimate in 2D.**

- The *discrete Beurling* estimate is an explicit form of the control that we need to deal with the harmonic measure H_δ near boundary points: it states that there exists constants $C, \alpha > 0$ such that for all z's

$$\left|H_\delta\left(z, B_\delta\right)\right| \leq C \left(\frac{\mathrm{d}\left(z, B_\delta\right)}{\mathrm{d}\left(z, C_\delta\right)}\right)^\alpha$$

- The strong discrete Beurling estimate (which we will not use in this class) gives that the optimal (i.e. biggest) universal α that can be chosen is $\alpha = \frac{1}{2}$.

2.7. **General Case.**

- In dimension greater than 2, there is no discrete Beurling estimate, so one must add conditions about the boundary in order not to have strange behavior.
- If, for instance, the boundary is smooth, one can argue that the neighborhood of each boundary point looks like a half-space and use Beurling-like arguments.

2.8. **Discrete Harnack Inequality.**

- Let us prove the 2D Harnack inequality: the result and the proof are essentially the same in all dimensions.
- Consider the discretization $\mathbb{D}_\delta$ of a disk $\mathbb{D} = \{|z| < r\}$ by $\delta\mathbb{Z}^2$ and identify the points of the plane with complex numbers.
- Let us show the following inequality: for any discrete harmonic function $f : \mathbb{D}_\delta \to \mathbb{R}$, we have

$$\left|f\left(i\delta\right) - f\left(-i\delta\right)\right| \leq \mathrm{Cst} \cdot \delta \cdot \max_{z \in \partial\mathbb{D}_\delta} \left|f\left(z\right)\right|.$$

- Similar arguments allow one to bound $\left|f\left(\delta\right) - f\left(0\right)\right|$ and $\left|f\left(i\delta\right) - f\left(0\right)\right|$.
- The idea is to represent $f\left(\pm i\delta\right)$ as $\mathbb{E}\left[f\left(X_{\tau^\pm}^{\pm i\delta}\right)\right]$ for SRWs starting from $\pm i\delta$ where $\tau^\pm$ are the hitting times of $\partial\mathbb{D}_\delta$ and compare the expectations.
- The idea to compare the expectations is to *couple* the SRWs $\left(X_n^{i\delta}\right)_{n\geq 0}$ and $\left(X_n^{-i\delta}\right)_{n\geq 0}$ in such a way that most of the time, they hit $\partial\mathbb{D}_\delta$ at the same place.
- The coupling is the following: sample $\left(X_n^{-i\delta}\right)_{n\geq 0}$ by taking the mirror image with respect to $\mathbb{R}$ of $\left(X_n^{i\delta}\right)_{n\geq 0}$ until the first time ς that $X_\varsigma^{i\delta} \in \mathbb{R}$, after which we set $X_n^{-i\delta} = X_n^{i\delta}$.
- It is easy to see that this is a coupling (i.e. that $X_n^{-i\delta}$ is a SRW) and that for this coupling

$$\left|f\left(i\delta\right) - f\left(-i\delta\right)\right| \leq \mathbb{P}\left\{\tau^+ < \varsigma\right\} \max_{z \in \partial\mathbb{D}_\delta} \left|f\left(z\right)\right|.$$

- We can bound $\mathbb{P}\left\{\tau^+ < \varsigma\right\}$, i.e. the chance that $X_n^{i\delta}$ hits $\partial\mathbb{D}_\delta$ before $\mathbb{R}$ by $\mathrm{Cst} \cdot \delta$.

2.9. **Uniqueness.**

- Now take $\delta \to 0$ and then $r \to 0$, we get a function on G, which is smooth, has boundary value 1 on B and boundary value 0 on $\partial G \setminus B$; is it unique?
- As usual, by linearity we just need to argue that we have a continuous maximum principle. This is the case since for harmonic functions we have the mean value property: $f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{i\theta})d\theta$.

3. The continuous Laplace Equation

3.1. **Holomorphic functions.**

- A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic at z if there exits $f'(z) \in \mathbb{C}$ such that for any $h \in \mathbb{C}$, $f(z+h) \simeq f(z) + f'(z)h + o(|z|)$.
- We can look at complex-valued functions $f : \mathbb{C} \to \mathbb{C}$ as functions $f : \mathbb{R}^2 \to \mathbb{R}^2$.
- For any $\epsilon \in \mathbb{R}$, $f(z+\epsilon) \simeq f(z) + \epsilon f'(z)$ and thus $\partial_x f = f'(z)$. Also $f(z+\epsilon i) \simeq f(z) + i\epsilon f'(z)$ and thus $\partial_y f = if'(z) = i\partial_x f$.
- Thus $\left(\partial_x + i\partial_y\right) f = 0$, or if we define $\partial_{\overline{z}} = \left(\partial_x + i\partial_y\right)$, the holomorphicity of f means that

$$\partial_{\overline{z}} f = 0.$$

- If we write $\partial_y f = i\partial_x f$ using the real and imaginary parts of f, we get the Riemann-Cauchy equations:
$$\begin{cases} \partial_x \Re(f) = \partial_y \Im(f), \\ \partial_y \Re(f) = -\partial_x \Im(f). \end{cases}$$
- Holomorphic functions are harmonic (the real and imaginary parts are harmonic):
  - First point of view: $\Delta = \partial_z \partial_{\overline{z}}$ where $\partial_z = \partial_x - i\partial_y$, thus since $\partial_{\overline{z}} f = 0$, we have $\Delta f = 0$.
  - Second point of view: we show $\Delta \Re(f) = 0$, similar proof holds for $\Im(f)$. Using Riemann-Cauchy equations,
$$\partial_x \partial_x \Re(f) = \partial_x \partial_y \Im(f) = \partial_y \partial_x \Im(f) = -\partial_y \partial_y \Re(f),$$
hence $\Delta \Re(f) = 0$.
- So we have a way to construct harmonic functions: just consider the real part of holomorphic functions.
- The link between holomorphic functions and harmonic functions is stronger in fact: any harmonic function on a simply connected domain is the real part of a holomorphic function (see the Exercise Sheet 6).
- If $f : \Omega \to \tilde{\Omega}$ is holomorphic and $g : \tilde{\Omega} \to \mathbb{R}$ is harmonic, then $g \circ f : \Omega \to \mathbb{R}$ is harmonic. (Simple computation)

3.2. **Poisson Kernel.**

- The goal of the section is to understand the Poisson kernel formula: if g is continuous on the unit circle $\partial\mathbb{D}$, where $\mathbb{D} := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\right\}$, then the unique solution of the continuous Laplace equation is given by
$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \right) g(e^{i\theta}) d\theta.$$
- How can we obtain this formula ?
- As for the discrete, we decompose in easier sub-problems: we want to find a function f which is harmonic and which is equal to 0 everywhere except in $e^{i\theta}$ where it should be a "Dirac".
- Such function should be the real part of an holomorphic function $\psi : \mathbb{D} \to \mathbb{C}$ such that $\Re\psi(e^{i\theta}) = 0$ for any θ, except in 1 where something special should happens.
- The function ψ sends then $\partial\mathbb{D}$ on the line $x = 0$: we are thus looking for an holomorphic function ψ which maps $\mathbb{D}$ on $\mathbb{H}_{\Re \ge 0} = \{z \in \mathbb{C}, \Re(z) \ge 0\}$, and which maps 1 on $\pm\infty$. It is given by
$$\psi(z) = C.\frac{z + 1}{1 - z}$$
where $C \in \mathbb{R}$.
- We define $P_1(z) = C.\Re \left(\frac{z+1}{1-z} \right)$. If we want to solve the similar problem but where the "Dirac" is at $e^{i\theta}$, we get $P_\theta(z) = P_1(ze^{-i\theta})$.
- By linear supperposition of the solutions, we get that the solution of the continuous Laplace equation where the boundary condition is given by g is
$$f(z) = \int_{-\pi}^{\pi} P_\theta(z) g(\theta) d\theta.$$
It remains to know which constant to take (i.e. what is C).
- When $g = 1$, the solution is constant equal to 1. This allows to fix the constant and we obtain
$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \right) g(e^{i\theta}) d\theta.$$

3.3. **The scaling limit of the exit probability of a RW.**

- Consider the unit disk $\mathbb{D} := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\right\}$ and discretize it by $\mathbb{D}_\delta$ with mesh $\delta \to 0$.
- What is the chance that a SRW from (x, y) exits $\mathbb{D}_\delta$ through the upper-right quadrant as $\delta \to 0$?
- We know that this probability is given by the discrete harmonic measure $H_\delta((x, y), I)$ where I is the upper-right quadrant.
- We know that $\lim_{\delta\to 0} H_\delta ((x, y), I)$ is the continuous harmonic measure $H((x, y), I)$. And by the last section, we have an explicit formula for it.
- Hence
$$\lim_{\delta\to 0} \mathbb{P}_z^{\mathbb{D}_\delta} [X_n \text{ exits } \mathbb{D}_\delta \text{ through the upper-right quadrant}] = \frac{1}{2\pi} \int_0^{\frac{\pi}{4}} \Re \left(\frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} \right) d\theta.$$

LATTICE MODELS (MA1, EPFL)

1. Uniform Spanning Tree

We consider a finite connected graph $\mathbb{G}$, with no double edge (i.e. two vertices are connected by zero or one edge), and no self-edge.

1.1. **Definitions.**

- A spanning tree T of a connected graph $\mathbb{G}$ is a subgraph of $\mathbb{G}$ that is a tree (connected, no cycle) and that contains all the vertices of $\mathbb{G}$. The set of spanning trees of $\mathbb{G}$ is denoted by $\mathrm{T}_{\mathbb{G}}$.
- There is a finite number of spanning trees of a finite graph. We can consider the uniform measure $\mathbb{P}[T] = \frac{1}{\#\mathrm{T}_{\mathbb{G}}}$.
  - Can we count the number of spanning trees?
  - Can we simulate a uniform spanning tree?

1.2. **Counting Spanning Trees: the matrix-tree theorem / Kirchhoff's theorem.**

- The number of spanning trees is related to the *Laplacian matrix*:

$$\begin{aligned} L_{v,v} &= \deg v, \\ L_{v,w} &= -1, \end{aligned}$$

  for $v \sim w$ and $v \neq w$.
- The Laplacian matrix is related to the matrix of the Laplacian Δ:

$$-\left(\delta_{v=w}\frac{1}{\deg v}\right)_{v,w} L = \Delta.$$

- Kirchhoff's theorem (Proof in Exercise Sheet 5): The number of covering trees of $\mathbb{G}$ is given by

$$\#\mathrm{T}_{\mathbb{G}} = \left|\det L^{i,j}\right| = (-1)^{i,j} L^{i,j},$$

  where $L^{i,j}$ is the Laplacian matrix where we have removed the i^{th} line and j^{th} column.
- If $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \ldots \leq \lambda_n$ are the eigenvalues of L,

$$\#\mathrm{T}_{\mathbb{G}} = \frac{1}{n}\prod_{i=2}^{n} \lambda_i.$$

  which is obtained by considering the linear term of the characteristic polynomial $P_L(X)$ of L in two different ways:
  - $P_L(X) = X\prod_{i=2}^{n}(X - \lambda_i)$: the linear tern is $(-1)^{n-1}\prod_{i=2}^{n}\lambda_i$.
  - $P_L(X) = \det(X.\mathrm{Id} - L)$: the linear term is $(-1)^{n-1}\sum_{i=1}^{n}\det(L^{i,i}) = (-1)^{n-1}n\#\mathrm{T}_{\mathbb{G}}$.
- The Kirchhoff's theorem can be generalized to the case where vertices can be connected by more than one edge: one has to define $\deg(v)$ as the number of edges which contain v, and $L_{v,w} = -\#\{\text{edges } \ v \leftrightarrow w\}$.

1.3. **Sampling Uniform Spanning Trees via counting.**

- Sampling each edge one by one recursively, by computing the probability that a given edge is selected in the spanning tree knowing that we have already decided that some other edges are in the tree or not.
- Enumerate the edges of $\mathbb{G}$: $e_1, \ldots, e_n$.
- The probability that $e_1 \in T$: $\mathbb{P}\left[e_1 \in T\right] = \frac{\#\{T|e \in T\}}{\#\mathrm{T}_{\mathbb{G}}}$.
- Bijections:
  - The set of spanning trees which contain e is in bijection with the set of spanning trees of the graph $\mathbb{G}^{\vee\{e\}}$ obtained by collapsing the edge e and glue together the two endpoints of e.
  - The set of spanning trees which do not contain e is in bijection with the set of spanning trees of the graph $\mathbb{G} \setminus \{e\}$ obtained by erasing the edge e in the graph $\mathbb{G}$.
- Thus, $\mathbb{P}\left[e_1 \in T\right] = \frac{\#\{T|e \in T\}}{\#\mathrm{T}_{\mathbb{G}}} = \frac{\det L^{1,1}_{\mathbb{G}^{\vee\{e_1\}}}}{\det L^{1,1}_{\mathbb{G}}}$ can be computed: we can randomly decide if e_1 will be in the spanning uniform tree.

- Once we have decide if e_1 will be in the uniform spanning tree, we have to decide if e_2 is in or not: if T is a uniform spanning tree,
  - the law of T knowing that $e_1 \in T$ is the law of a uniform spanning tree on $\mathbb{G}^{\vee\{e\}}$.
  - the law of T knowing that $e_2 \notin T$ is the law of a uniform spanning tree on $\mathbb{G} \setminus \{e\}$.
- We decide that $e_2 \in T$ with the following probabilities:
  - if $e_1 \in T$: $\mathbb{P}\left[e_2 \in T \mid e_1 \in T\right] = \frac{\det L^{1,1}_{\mathbb{G}^{\vee\{e_1,e_2\}}}}{\det L^{1,1}_{\mathbb{G}^{\vee\{e_1\}}}}$
  - if $e_1 \notin T$: $\mathbb{P}\left[e_2 \in T \mid e_1 \notin T\right] = \frac{\det L^{1,1}_{(\mathbb{G}\setminus\{e_1\})^{\vee\{e_2\}}}}{\det L^{1,1}_{\mathbb{G}\setminus\{e_1\}}}$
- and so on and so forth.
- Trivially works: we are sampling a uniform spanning tree, but very costly way to sample it !

1.4. **Sampling Uniform Tree using Random Walks: the Aldous-Broder algorithm.**

- The set of edges of the tree you are going to built is initially empy $E_T = \{\}$.
- Choose an initial vertex X_0 and launch a simple random walk on $\mathbb{G}$ (i.e. jump uniformly to the neighbours independently from the past).
- While all vertices have not been visited:
  - if X_n is a vertex which has not been yet visited, remember the edge $\{X_{n-1}, X_n\}$:
$$E_T = E_T \cup \{\{X_{n-1}, X_n\}\}$$
  - if X_n is a vertex which has already been visited, do not record the edge.
- Yields a better algorithm to sample uniform spanning trees. We will not prove that the algorithm yields a uniform spanning tree, we will consider instead the Wilson's Algorithm, based on the notion of Loop-Erased Random Walk.

1.5. **Sampling Uniform Tree using Loop-Erased Random Walks: the Wilson's algorithm.**

1.5.1. *Loop-Erased Random Walk.*

- A loop-erased random walk (LERW) is built by applying a loop-erasure procedure to a simple random walk.
- The so-called chronological *loop-erasure procedure* is as follows: whenever the simple random walk closes a loop, i.e. arrives at time n to a vertex where it has come before at time t_n, the list of points visited between times t_n and n is erased.

1.5.2. *Wilson's Algorithm.*

- Wilson's algorithm is a very efficient way to uniformly sample a spanning tree by adding branches using LERW.
- Consider an enumeration of the vertices $x_0, \dots, x_n$ where x_0 will be the root of the tree.
- The idea is to choose a root x_0 and to construct a growing family of trees $\{x_0\} = T_0 \subset T_1 \subset \cdots \subset T_k$ such that $T_{i+1} \setminus T_i$ is a LERW from an arbitrary vertex of $x_i \in G \setminus T_i$ stopped upon hitting T_i and stopping when we get a spanning tree.
- Obviously, we get a spanning tree with this method. The question is: why is the measure uniform?
- The idea is to construct a probability space made of 'stacks of arrows' that generates the tree and the loops that were erased in the LERW
- This will show that the measure is uniform and the tree is actually independent of the choices of $x_0, \dots, x_r$.

1.5.3. *Stack of Arrows.*

- To each vertex $x \in G \setminus \{x_0\}$, associate an infinite stack of random 'arrows' each of which points to a neighbor of x, uniformly, independently of each other.
- These arrows can be used to sample a SRW from any vertex $x \in G \setminus \{x_0\}$, stopped upon hitting x_0: when we are at a vertex, jump to the neighbor pointed by the arrow at the top of the stack and remove this arrow.
- If we look at the arrows at the top of the stacks, they form a number of cycles, plus a tree pointing to the root x_0.
- We can take the arrows of a cycle at the top of the stack and remove it: we call this a 'cycle removal' procedure. The Wilson algorithm is a way to explore the stack of arrows in order to remove cycles.
- If we keep removing cycles until we don't have any (this will happen with probability one), we get a tree.

- What we will show: the order in which we remove the cycles is irrelevant and the tree that we get is sampled with uniform probability.

1.5.4. *Cycle Lemma.*

- Lemma : if two people have two different ways to remove cycles in the stack of arrows, they end up removing exactly the same cycles.
- Let $C_1, \dots, C_n$ be a (ordered) sequence of cycles which can be popped in order to obtain a tree. Let $D_1, \dots, D_m$ be an other sequence which can be popped in order to obtain a tree. We show by induction that $n = m$, and there exists a permutation σ such that $C_i = D_{\sigma(i)}$.
- If $n = 0$, that is trivial.
- If $n > 0$, then at least one of the cycles $D_1, \dots, D_m$ should intersect C_1 (if not, C_1 would still be there after removing $D_1, \dots, D_m$ and thus we would not obtain a tree after removing them). Consider the minimal $i \in \{1, \dots, m\}$ such that D_i intersects C_1.
- Consider v a vertex in the intersection $D_i \cap C_1$: because of the minimality of the index i, the arrow in D_i and C_1 going from v must be the same (because they are taking the first arrow from the stack associated to v).
- But then the next vertex (following the arrow) is also the same for C_1 and D_i, and so on and so forth. So we get $C_1 = D_i$, in particular D_i was on the top of the stack of arrows.
- Thus, instead of removing $D_1, \dots, D_m$, one could have removed $D_i, D_1, \dots, D_{i-1}, D_{i+1}, \dots, D_m$. And since $C_1 = D_i$, after we remove D_i or C_1, we have a new stack of arrows, and removing $C_2, \dots, C_{n-1}$ or $D_1, \dots, D_{i-1}, D_{i+1}, \dots, D_m$ both allow to obtain a tree. By induction $n - 1 = m - 1$ and the $D_i s$ are the same cycles than the C_is up to a permutation.

1.5.5. *Proof of Wilson's algorithm.*

- Once we understand the picture (the stack of arrows yields a collection of loops and a tree just below), it is fairly easy to show that the tree that we get is uniform.
- Let us look at a possible 'history', i.e. a set of loops sitting on top of a tree. It has the same probability as an alternate history with the same set of loops on top of any different tree.
- To compute the probability of a tree, we sum over all the possible 'histories' leading to it: we deduce from the above remark that each tree has the same probability.

LATTICE MODELS (MA1, EPFL)

1. Percolation: Cardy's Formula For Crossing Probabilities

1.1. **Different kind of percolations.**

- We consider a lattice graph (e.g. $\mathbb{Z}^2$, triangular lattice, honeycomb lattice, ...), with sets of vertices $\mathcal{V}$, edges $\mathcal{E}$ and faces $\mathcal{F}$.
- For $\mathcal{S} = \mathcal{V}, \mathcal{E}$ or $\mathcal{F}$ we can consider a model of percolation associated with it. Each element $s \in \mathcal{S}$ will be either "open" or "closed" (or 1/0 or white/black) independently from the other $s' \in \mathcal{S}$. Let $p \in [0, 1]$ be the probability of $s \in \mathcal{S}$ is open: we are considering a family of independent Bernoulli random variables $(X_s)_{s\in\mathcal{S}}$ of parameter p.
- If $\mathcal{S} = \mathcal{V}$, this is a site percolation, if $\mathcal{S} = \mathcal{E}$, this is an edge percolation, if $\mathcal{S} = \mathcal{F}$, this is a face percolation.
- Using the dual lattice, face percolation is equivalent to site percolation on the dual lattice (e.g. face percolation on the hexagonal lattice is the same as site percolation on the triangular lattice). So there exists essentially two kinds of percolations: edge and face percolations. In the exercises, you will study the edge percolation mainly on $\mathbb{Z}^2$, in the lesson we will study the face percolation on the honeycomb lattice.

1.2. **Natural questions: phase transition and crossing probabilities.**

- Percolation is originally a model of a porous medium: we can think that black hexagons are filled with matter, while white ones are empty.
- It is natural to ask about connectivity questions for that medium:
  - is there a path made of white faces connecting the origin to infinity ?
  - is there an infinite connected component of white faces ?
  - is there a path made of white hexagons joining two sets ?
- Phase transition: we can consider $\mathcal{O}(p) = \mathbb{P}_p\,[\exists$ an infinite connected component of white face$]$ and $\theta(p) = \mathbb{P}_p[0 \leftrightsquigarrow \infty]$. We have:
  - $\mathcal{O}(p)$ and $\theta(p)$ are increasing in p (one can couple the percolations with any parameter by considering a family of independent uniform $[0, 1]$ random variables $(U_f)_{f\in\mathcal{F}}$: a face percolation with parameter p is obtained by considering $(U_f \leq p)_{f\in\mathcal{F}}$. In that case, if a face is open/white/1 for p, then it remains open/white/1 for $p' \geq p$)
  - $\mathcal{O}(0) = 0 = \theta(0)$ and $\mathcal{O}(1) = 1 = \theta(1)$.
  - $\mathcal{O}(p) = 0$ or $\mathcal{O}(p) = 1$ (Kolmogorov's zero-one law since one can enumerate the faces and the event that there exists an infinite connected component of white face does not depend on the values of any first values $X_{f_1}, \dots, X_{f_n}$).
  - $\theta(p) > 0$ if and only if $\mathcal{O}(p) = 1$. (see Exercise 2, Sheet 8)
  - $\theta(p)$ is right continuous. (see Exercise 2, Sheet 8)
  - Using the above properties, we see that there exists p_c such that:
    * $\mathcal{O}(p) = 0$ for $p \leq p_c$, $\mathcal{O}(p) = 1$ for $p > p_c$,
    * $\theta(p) = 0$ for $p \leq p_c$, $\theta(p) > 0$ for $p > p_c$.
  - This is the critical probability of the percolation: p_c depends on the type of percolation (edge/face percolation and the kind lattice): for example, $p_c = \frac{1}{2}$ for the edge percolation on $\mathbb{Z}^2$, $p_c = \frac{1}{2}$ for the face percolation on the honeycomb lattice [Kesten1980], but $p_c = 1 - 2\sin(\frac{\pi}{18}) \sim 0.6527$ for the edge percolation on the hexagonal lattice.
- If we are at the critical probability, we can consider the "continuous" limit, i.e. we consider the percolation on the lattice with mesh δ, and ask ourselves if we can compute the probability, as $\delta \to 0$, that there exists a path made of white hexagons joining two sets. For example, we fix a rectangle $[0, 1] \times [0, N]$, we discretize it using a hexagonal lattice, and we ask ourselves what is

$$\lim_{\delta\to 0} \mathbb{P}_{\frac{1}{2}}\left[\{0\} \times [0, 1] \leftrightsquigarrow_w \{1\} \times [0, 1]\right]$$

where $\leftrightsquigarrow_w$ means that there exists a white path connecting the two sets (which are the lower and upper part of the rectangle). The answer is given by Cardy's formula.

- From now on, we consider the critical face percolation on the honeycomb lattice (i.e. $p = \frac{1}{2}$) where the edges are parallel to $1, \tau = e^{2\pi i/3}, \tau^2 = e^{-2i\frac{\pi}{3}}$.

1.3. Cardy's formula.

- Consider a quad $(\Omega, a_1, a_2, a_3, a_4)$, i.e. Ω is a Jordan domain (i.e. $\partial\Omega$ is a closed simple curve) and four points $a_1, a_2, a_3, a_4 \in \partial\Omega$ in counterclockwise (or c.c.w.) order, a discretization $(\Omega_\delta)_{\delta>0}$ by a honeycomb domain of mesh size δ, and identify each point a_1, a_2, a_3, a_4 with the closest boundary vertex.
- Crossing probabilities: what is the chance $\mathbb{P}_{\Omega_\delta}\{[a_1a_2] \leftrightsquigarrow_w [a_3a_4]\}$ that there is a path of white hexagons linking $[a_1a_2]$ to $[a_3a_4]$?
- Two main properties of the limiting crossing probabilities as $\delta \to 0$:
  - *Conformal invariance of crossing probabilities* (Smirnov, 2001): if $(\Omega, a_1, a_2, a_3, a_4)$ and $\left(\tilde{\Omega}, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4\right)$ are conformally equivalent (i.e. there exists a conformal map $\Omega \to \tilde{\Omega}$ with $a_1, a_2, a_3, a_4 \mapsto \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4$), then
  $$\lim_{\delta\to 0} \mathbb{P}_{\Omega_\delta}\{[a_1a_2] \leftrightsquigarrow_w [a_3a_4]\} = \lim_{\delta\to 0} \mathbb{P}_{\tilde{\Omega}_\delta}\{[\tilde{a}_1\tilde{a}_2] \leftrightsquigarrow_w [\tilde{a}_3\tilde{a}_4]\}.$$
  - *Explicit formula* (Cardy's formula) *for the equilateral triangle*: if Δ is an equilateral triangle with vertices a, b, c, then
  $$\lim_{\delta\to 0} \mathbb{P}_{\Delta_\delta}([ab] \leftrightsquigarrow [cd]) = \frac{|c-d|}{|c-a|}.$$
- How can we compute the limiting crossing probabilities for any quad $(\Omega, a_1, a_2, a_3, a_4)$? Using Riemann's mapping theorem.

1.4. Riemann's mapping theorem.

- Let Ω and $\tilde{\Omega}$ be two Jordan domains (i.e. $\partial\Omega$ and $\partial\tilde{\Omega}$ are closed simple curves), and let $a_1, a_2, a_3 \in \partial\Omega$ and $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \in \partial\tilde{\Omega}$ be distinct boundary points, in c.c.w. order.
- There exists a *unique* conformal mapping (i.e. holomorphic + bijective) $\varphi : \Omega \to \tilde{\Omega}$ with $a_1, a_2, a_3 \mapsto \tilde{a}_1, \tilde{a}_2, \tilde{a}_3$.
- Remark: this statement can be generalized to arbitrary simply-connected domains, provided the boundary points are replaced by prime ends.

1.5. Cardy's formula and main statement.

- For a Jordan domain (Ω, a_1, a_2, a_3), there exists a unique conformal mapping φ from Ω to the equilateral triangle $\triangle$ with vertices $1, \pm\frac{\sqrt{3}}{3}i$, with $a_1, a_2, a_3 \mapsto a := 1, b := \frac{\sqrt{3}}{3}i, c := -\frac{\sqrt{3}}{3}i$.
- Using the two main properties of the limiting crossing probabilities and Riemann's mapping theorem, we obtain Carleson's formulation of Cardy's formula [Smirnov 2001]:
$$\lim_{\delta\to 0} \mathbb{P}_{\Omega_\delta}\{[a_1a_2] \leftrightsquigarrow_w [a_3a_4]\} = \mathfrak{Re}\left(\varphi\left(a_4\right)\right)$$
(apply the conformal invariance, Cardy's formula for the equilateral triangle Δ, and Thalès theorem.)
- Carleson's formulation of the limiting crossing probabilities is equivalent to the two main properties of the limiting crossing probabilities given above:
  - We obtained it from the two main properties.
  - If we assume instead Carleson's formulation, Cardy's formula is a direct consequence, and by te uniqueness of the conformal mapping $\varphi : \Omega \to \Delta$, we obtain conformal invariance. Indeed, if $(\Omega, a_1, a_2, a_3, a_4)$ and $\left(\tilde{\Omega}, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4\right)$ are conformally equivalent (i.e. there exists a conformal map $\phi : \Omega \to \tilde{\Omega}$ with $a_1, a_2, a_3, a_4 \mapsto \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4$), and if $\tilde{\varphi}$ is the unique conformal map $\tilde{\Omega} \to \Delta$ which sends $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$ on a, b and c, then $\tilde{\varphi} \circ \phi$ is the unique conformal map $\Omega \to \Delta$ which sends a_1, a_2, a_3 on a, b and c. Thus:
  $$\lim_{\delta\to 0} \mathbb{P}_{\Omega_\delta}\{[a_1a_2] \leftrightsquigarrow_w [a_3a_4]\} = \mathfrak{Re}(\tilde{\varphi} \circ \phi(a_4)) = \mathfrak{Re}(\tilde{\varphi}(\tilde{a}_4)) = \lim_{\delta\to 0} \mathbb{P}_{\tilde{\Omega}_\delta}\{[\tilde{a}_1\tilde{a}_2] \leftrightsquigarrow_w [\tilde{a}_3\tilde{a}_4]\}$$
  which proves the conformal invariance.
- To prove Cardy's formula, we are going to use the same strategy that we used in order to show the convergence of the discrete harmonic measure to the continuous harmonic measure. Yet, there are two problems:
  - Cardy's formula is about the convergence of functions on the boundary $[a_3, a_1]$,

  – $\Re(\varphi)$ can be characterized by the fact that it is the real part of the unique conformal map $\Omega \to \Delta$ which sends a_1, b_1, c_1 on a, b, c, but can we not use directly φ ?
- For the first point, we extend the observable $\mathbb{P}_{\Omega_\delta}\{[a_1a_2] \leftrightsquigarrow_w [a_3z]\}$ from the boundary $z \in [a_3, a_1]$ to the bulk $z \in \overline{\Omega}$. In order to do so, we consider $A_1 := [a_2a_3]$ and

$$H_\delta^1(z) := \mathbb{P}_{\Omega_\delta}\{a_1 \text{ and } z \text{ are separated from } A_1 \text{ by a white path}\}.$$

that we write also as

$$H_\delta^1(z) := \mathbb{P}_{\Omega_\delta}\{\{a_1, z\} \,||_w\, A_1\}.$$

Note that when $z \in [a_3, a_1]$ then $H_\delta^1(z) = \mathbb{P}_{\Omega_\delta}\{[a_1a_2] \leftrightsquigarrow_w [a_3z]\}$.
- For the second point, we consider now

$$H_\delta^\mu(z) := \mathbb{P}_{\Omega_\delta}[Q^\mu(z)]$$

where

$$Q^\mu(z) := \{\{a_\mu, z\} \,||_w\, A_\mu\}$$

for $\mu \in \{1, 2, 3\}$, where $A_1 := [a_2a_3]$, $A_2 := [a_3a_1]$, and $A_3 := [a_1a_2]$, and we set

$$\varphi_\delta := H_\delta^1 + \frac{\sqrt{3}}{3} i H_\delta^2 - \frac{\sqrt{3}}{3} i H_\delta^3.$$

- **Theorem:** When $\delta \to 0$, the function φ_δ converges to the unique conformal map $\varphi : \Omega \to \Delta$ which sends a_1, b_1, c_1 on a, b, c:

$$\varphi_\delta \underset{\delta \to 0}{\longrightarrow} \varphi.$$

- Since $\mathbb{P}_{\Omega_\delta}(\{a_1, a_4\} \,||_w\, A_1) = H_\delta^1(a_4) = \Re(\varphi_\delta(a_4))$, we get Carleson's formulation of Cardy's formula: $\lim_{\delta \to 0} \mathbb{P}_{\Omega_\delta}(\{a_1, a_4\} \,||_w\, A_1) = \Re(\varphi(a_4))$.

1.6. **Strategy of the proof.**

- *Precompactness:* we show that $(\varphi_\delta)_{\delta > 0}$ is uniformly equicontinuous on $\overline{\Omega}$ by showing that it is uniformly Hölder continuous. Since they are uniformly bounded, we can apply the Arzelà-Ascoli theorem: from any subsequence, we can extract a sub-subsequence that converges in uniform norm. If we show that the limit does not depend on the sub-subsequence considered and is equal to φ, we will have shown that $\varphi_\delta \to \varphi$. Let $\boldsymbol{\varphi}$ be a subsequential scaling limit, i.e. $\varphi_{\delta_n} \to_{n \to \infty} \boldsymbol{\varphi}$.
- *Boundary conditions:* we show $\boldsymbol{\varphi}$ is a homeomorphism $\partial\Omega \to \partial\triangle$ with $a_1, a_2, a_3 \mapsto 1, \frac{\sqrt{3}}{3} i, -\frac{\sqrt{3}}{3} i$.
- *Analyticity:* we show $\boldsymbol{\varphi}$ is analytic. This will follow from approximate discrete Cauchy-Riemann relations and Morera's criterion.
- *Bijection:* To show that $\boldsymbol{\varphi} : \Omega \to \triangle$ is a bijection inside Ω, we use the argument principle, take $w \in \mathbb{C}$: we want to show that $\varphi(z) - w$ has a single zero in Ω iff $w \in \triangle$:

$$\#\{\text{zeros of } \varphi(z) - w\} = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{\varphi'(z)}{\varphi(z) - w} \mathrm{d}z = \frac{1}{2\pi i} \oint_{\partial\triangle} \frac{1}{\zeta - w} \mathrm{d}\zeta = \mathbf{1}_\triangle(w).$$

1.7. **A priori estimates.** Like for the random walks, we first need some simple a priori estimates for the precompactness part.

1.7.1. *Symmetry, Self-Duality.*

- Consider the square $S = [0, 1] \times [0, i]$, and discretize S by a symmetric honeycomb domain S_δ, with $\delta > 0$ small.
- We have $\lim_{\delta \to 0} \mathbb{P}_{S_\delta}\{[0, i] \leftrightsquigarrow_w [1, 1 + i]\} = \frac{1}{2}$: either we have a horizontal white path or a vertical black path crossing, but since $p = 1/2$, both events have the same probability.

1.7.2. *FKG inequality.* In order to lower bound uniformly in δ probabilities such as $\mathbb{P}_{R_\delta}\{[0, i] \leftrightsquigarrow_w [2, 2 + i]\}$, we need to paste white paths together. For that, we need the Fortuin-Kasteleyn-Ginibre (FKG) inequality:

- An event $\mathcal{A}$ is increasing if for any family $(w_f)_{f \in \mathcal{F}} \in \{0, 1\}^{\mathcal{F}}$ and $(\tilde{w}_f)_{f \in \mathcal{F}} \in \{0, 1\}^{\mathcal{F}}$, such that for any face f, $w_f \leq \tilde{w}_f$, we have

$$(w_f)_{f \in \mathcal{F}} \in \mathcal{A} \implies (\tilde{w}_f)_{f \in \mathcal{F}} \in \mathcal{A}$$

- For example, the event "there is a white path from here to here" is an increasing event.
- FKG inequality: For $\mathcal{A}$ and $\mathcal{B}$ two increasing events, then for the percolation with parameter p, we have $\mathbb{P}_p(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p(\mathcal{A}) \mathbb{P}_p(\mathcal{B})$, or equivalently $\mathbb{P}_p(\mathcal{A} | \mathcal{B}) \geq \mathbb{P}_p(\mathcal{A})$.

- The latter is intuitive, because for exemple the existence of a white path somewhere can only increase the chances of seeing a white path elsewhere.
- The inequality is the same if they are decreasing events, and in the opposite direction if one is increasing and the other decreasing.
- Proof is by induction on the number of faces:
  - For the first step, let us show a more general version: if μ is a measure on $[0,1]$ and f and g are increasing, then
$$\int f(x)g(x)d\mu(x) \geq \int f(x)d\mu(x) \int g(y)d\mu(y).$$
This implies the FKG inequality with one face: consider $f = 1_{\mathcal{A}}$ and $g = 1_{\mathcal{B}}$ and $\mu = (1-p)\delta_0 + p\delta_1$. How to show that ? Using the fact that both f and g are increasing, we have:
$$\int (f(y) - f(x))(g(y) - g(x))d\mu(y)d\mu(x) \geq 0$$
and if we expand, we get FKG.
  - Then, assume we showed the FKG inequality for n faces and assume that the set of faces $\mathcal{F}$ has cardinal $n+1$. Let X and Y be two functions on $\{0,1\}^{\mathcal{F}}$ which are increasing (i.e. if for any face f, $w_f \leq \tilde{w}_f$, then $X(w) \leq X(\tilde{w})$ and $Y(w) \leq Y(\tilde{w})$). We show that
$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X]\,\mathbb{E}_p[Y]$$
which implies the FKG inequality for the face percolation on $\mathcal{F}$ (consider $X = 1_{\mathcal{A}}$ and $Y = 1_{\mathcal{B}}$). We enumerate the faces, denote by $p(0) = 1-p$ and $p(1) = p$:
$$\begin{aligned}\mathbb{E}_p[XY] &= \sum_{w_i \in \{0,1\}} X(w_1,\dots,w_{n+1})Y(w_1,\dots,w_{n+1}) \prod_{i=1}^{n+1} p(w_i) \\ &= \sum_{w_{n+1}\in\{0,1\}} p(w_{n+1}) \sum_{w_1,\dots,w_n\in\{0,1\}} X(w_1,\dots,w_{n+1})Y(w_1,\dots,w_{n+1}) \prod_{i=1}^{n} p(w_i).\end{aligned}$$
The term $\sum_{w_1,\dots,w_n\in\{0,1\}} X(w_1,\dots,w_{n+1})Y(w_1,\dots,w_{n+1}) \prod_{i=1}^{n} p(w_i)$ is simply
$$\mathbb{E}_p[X(\dots,w_{n+1})Y(\dots,w_{n+1})]$$
where the expectation is with respect to a percolation on the first n faces with parameter p. Since we assumed that the FKG inequality holds for n faces, and since X and Y are increasing, we have $\mathbb{E}_p[X(\dots,w_{n+1})Y(\dots,w_{n+1})] \geq \mathbb{E}_p[X(\dots,w_{n+1})]\mathbb{E}_p[Y(\dots,w_{n+1})]$. We denote:
$$\hat{X}(w_{n+1}) := \mathbb{E}_p[X(\dots,w_{n+1})], \quad \hat{Y}(w_{n+1}) := \mathbb{E}_p[Y(\dots,w_{n+1})]$$
thus
$$\mathbb{E}_p[XY] \geq \sum_{w_{n+1}\in\{0,1\}} p(w_{n+1})\hat{X}(w_{n+1})\hat{Y}(w_{n+1})$$
and we recognize in the right-hand side: $\mathbb{E}\left[\hat{X}(w)\hat{Y}(w)\right]$ where w is a Bernoulli of parameter p. Since X (resp. Y) is increasing, $\hat{X}$ (resp. $\hat{Y}$) is increasing: we can apply FKG inequality: $\mathbb{E}\left[\hat{X}(w)\hat{Y}(w)\right] \geq \mathbb{E}\left[\hat{X}(w)\right]\mathbb{E}\left[\hat{Y}(w)\right]$. If we expand the right-hand side, we get that $\mathbb{E}\left[\hat{X}(w)\right]$ is equal to:
$$\begin{aligned}\sum_{w_{n+1}\in\{0,1\}} p(w_{n+1})\hat{X}(w_{n+1}) &= \sum_{w_{n+1}\in\{0,1\}} p(w_{n+1}) \sum_{w_1,\dots,w_n\in\{0,1\}} X(w_1,\dots,w_{n+1}) \prod_{i=1}^{n} p(w_i) \\ &= \sum_{w_{n+1}\in\{0,1\}} \sum_{w_1,\dots,w_n\in\{0,1\}} X(w_1,\dots,w_{n+1}) \prod_{i=1}^{n+1} p(w_i) = \mathbb{E}_p[X]\end{aligned}$$
and similarly for $\hat{Y}$. Hence
$$\mathbb{E}_p[XY] \geq \mathbb{E}\left[\hat{X}(w)\hat{Y}(w)\right] \geq \mathbb{E}_p[X]\mathbb{E}_p[Y]$$
and thus we have proven the FKG inequality for $n+1$ faces.

1.7.3. *RSW estimate.*

- Consider the rectangle $R = [0, 2] \times [0, i]$ and a symmetric discretization R_δ. We would like to show the Russo-Seymour-Welsh (RSW) estimate: we have $\mathbb{P}_{R_\delta}\{[0, i] \leftrightsquigarrow_w [2, 2 + i]\} \geq \frac{1}{16}$.
- With the FKG inequality, we can construct a crossing by pasting paths, and thus we can prove RSW:
  - With probability $\frac{1}{2}$, there is a path $\gamma : [0, i] \leftrightsquigarrow [1, 1 + i]$, made only of white hexagons, take the lowest such path possible. Whatever is above it is independent percolation.
  - Let $\tilde{\gamma}$ be the reflection of γ with respect to the line $1 + i\mathbb{R}$ and let D_δ be the connected component $R_\delta \setminus (\gamma \cup \tilde{\gamma})$ lying above $\gamma \cup \tilde{\gamma}$ and intersecting the line $1 + i\mathbb{R}$.
  - With probability $\frac{1}{2}$, there is a path $\lambda \subset D_\delta$ linking the left and bottom-left part of ∂D_δ (i.e. the part on the line $i\mathbb{R}$ and γ) to the top-right one (i.e $[1 + i, 2 + i]$), again by symmetry. Joining γ and λ yields a white path $[0, i] \leftrightsquigarrow [1 + i, 2 + i]$.
  - So, we have $\mathbb{P}_{R_\delta}(\mathcal{A}) \geq \frac{1}{4}$, where $\mathcal{A} := \{[0, i] \leftrightsquigarrow_w [1 + i, 2 + i]\}$. By symmetry, we have $\mathbb{P}_{R_\delta}(\mathcal{B}) \geq \frac{1}{4}$, where $\mathcal{B} := \{[2, 2 + i] \leftrightsquigarrow_w [i, 1 + i]\}$.
  - If both $\mathcal{A}$ and $\mathcal{B}$ occur, there is a white path $[0, i] \leftrightsquigarrow [2, 2 + i]$, i.e.

$$\mathbb{P}_{R_\delta}\{[0, i] \leftrightsquigarrow_w [2, 2 + i]\} \geq \mathbb{P}(\mathcal{A} \cap \mathcal{B})$$

    and by FKG, $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geq \frac{1}{16}$, which is the desired result.
- More generally, with FKG and RSW, the probability of a crossing in the discretizations of rectangles $[0, L] \times [0, i]$ are uniformly bounded from below with respect to $\delta > 0$ (and from above by duality). Indeed, the probability of crossing is lower bounded by the probability of:

$$\mathcal{A}_{0,2} \cap \mathcal{A}_{1,3} \cap ... \cap \mathcal{A}_{L-2,L} \cap \mathcal{B}_1 \cap \ldots \cap \mathcal{B}_{L-2}$$

  where $\mathcal{A}_{k,k+2}$ is the event $[k, k + i] \leftrightsquigarrow_w [k + 2, k + i]$ and $\mathcal{B}_k$ is the even $[k, k + 1] \leftrightsquigarrow_w [k + i, k + 1 + i]$. The probability of each event can be uniformly lower bounded, hence by FKG, we can lower bound from below $\mathbb{P}_{\mathcal{R}_\delta}([0, i] \leftrightsquigarrow_w [L, L + i])$ where $\mathcal{R}_\delta$ is a dicretization of $[0, L] \times [0, i]$.

1.7.4. *Annulus crossing estimate.*

- By FKG and RSW, we get that the probability of a white loop in a "square" annulus of inner and outer radii 1 and 2 is uniformly bounded from below with respect to δ (we paste again some rectangles with ratio of the lengths $L/\ell = 4$).
- For $r, R > 0$ consider the discretization A_δ of a "square" annulus of inner radius r and outer radius R. The probability that a black path links the inner circle to the outer circle is bounded by $C\left(\frac{r}{R}\right)^\alpha$, for universal $\alpha, C > 0$.
  - To prove that, we decompose A_δ into $k := \lfloor \log_2\left(\frac{R}{r}\right) \rfloor$ concentric annuli A_δ^j of inner and outer radii $2^{j-1}r$ and $2^j r$ for $j = 1, \ldots, k$
  - For each annulus, there is a uniformly positive chance of a white crossing in it.
  - Thus

$$\begin{aligned}
\mathbb{P}[\text{inner circle} \leftrightsquigarrow_b \text{outer circle}] &= 1 - \mathbb{P}[\text{white circle inside the annulus}] \\
&\leq 1 - \mathbb{P}\left[\exists j, \exists \text{white circle inside } A_\delta^j\right] \\
&= \mathbb{P}\left[\forall j, \nexists \text{ white circle inside } A_\delta^j\right] \\
&\leq (1 - c)^k
\end{aligned}$$

    where $c > 0$ is given by the previous annulus crossing estimate and we used in the last inequality the independence of the percolations in the disjoint area A_δ^j. Since $k := \lfloor \log_2\left(\frac{R}{r}\right) \rfloor$, we obtain $(1 - c)^k \leq C\left(\frac{r}{R}\right)^\alpha$ with $C, \alpha > 0$.

1.8. **Precompactness.**

- To show precompactness, we show that $\left(H_\delta^1\right)_\delta$ is uniformly Hölder continuous (the same reasoning applies to H_δ^2 and H_δ^3)
  - There exists $C > 0$ and $\alpha > 0$ which do not depend on δ, such that for any $x, y \in \Omega_\delta$, $\left|H_\delta^1(x) - H_\delta^1(y)\right| \leq C\mathrm{d}_\Omega(x, y)^\alpha$, where $\mathrm{d}_\Omega(x, y)$ is the length of the shortest path from x to y in $\overline{\Omega}$
- How to prove this? Let us assume x and y close (otherwise, there is nothing to prove).

  - We have that (writing $\|_w$ for 'there is a white path separating')
$$\begin{aligned} H_\delta^1(x) - H_\delta^1(y) &= \mathbb{P}_{\Omega_\delta}(\{a_1, x\|_w A_1\}) - \mathbb{P}_{\Omega_\delta}(\{a_1, y\|_w A_1\}) \\ &= \mathbb{P}_{\Omega_\delta}(\{a_1, x\|_w A_1\} \setminus \{a_1, y\|_w A_1\}) - \mathbb{P}(\{a_1, y\|_w A_1\} \setminus \{a_1, x\|_w A_1\}) \end{aligned}$$
    where we used the fact that $\mathbb{P}[A] - \mathbb{P}[B] = \mathbb{P}[A \setminus B] - \mathbb{P}[B \setminus A]$.
  - Let us study the probability of the event $E_1(x, y) = \{a_1, x\|_w A_1\} \setminus \{a_1, y\|_w A_1\}$.
    * We see that the occurrence of E_1 implies the existence of a white path from A_2 to A_3 passing between x and y, and a black path from A_1 to A_1 separating x and y.
    * In turn, each such path implies that a white or a black path goes from a 'microscopic' circle (i.e. of radius $\mathrm{d}_\Omega(x, y)$) to a macroscopic circle (i.e. of radius $\mathrm{dist}(\{x, y\}, A_j)$ for $j = 1, 2, 3$).
    * By topology argument, at least one of the macroscopic circles is 'big' (i.e. greater than a uniform ϵ), and we can bound the probability that this path exists by the application of RSW above, by the annulus crossing estimate.
  - Hence, we get $\mathbb{P}(E_1(x, y)) \leq C\mathrm{d}_\Omega(x, y)^\alpha$, where $C, \alpha > 0$ do not depend on δ, and by symmetry, we deduce the uniform Hölder-continuity of H_δ^1.
- To extract converging subsequences, we extend φ_δ into a continuous function (by piecewise affine interpolation for instance), and use Arzelà-Ascoli: obviously $(\varphi_\delta)_\delta$ is bounded and equicontinuous.
- From now on, we will assume that $\boldsymbol{\varphi}$ is a subsequential scaling limit i.e. $\boldsymbol{\varphi} = \lim_{\delta_n \to 0} \varphi_{\delta_n}$, with $\varphi_\delta = H_\delta^1 + i\frac{\sqrt{3}}{3}H_\delta^2 - i\frac{\sqrt{3}}{3}H_\delta^3$. We also denote by $H^\mu = \lim_{\delta_n \to 0} H_{\delta_n}^\mu$.
- It remains to show that $\boldsymbol{\varphi}$ is the unique conformal map $\varphi : \Omega \to \Delta$ which sends a_1, a_2 and a_3 on a, b, and c.

1.9. **Boundary conditions.**
- We want to prove that $\boldsymbol{\varphi}$ is a homeomorphism $\boldsymbol{\varphi} : \partial\Omega \to \partial\Delta$, so we want to prove that it is a homeomorphism $\boldsymbol{\varphi} : A_1 \to \left[\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}\right]$, $\boldsymbol{\varphi} : A_2 \to \left[-\frac{i\sqrt{3}}{3}, 1\right]$, $\boldsymbol{\varphi} : A_3 \to \left[1, \frac{i\sqrt{3}}{3}\right]$. By symmetry, it is enough to prove the first assertion.
- To prove that $\boldsymbol{\varphi} : A_1 \to \left[\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}\right]$ is a homeomorphism, we should prove that
  (1) For any $z \in A_1$, $\Re\boldsymbol{\varphi}(z) = 0$, i.e. $H^1(z) = 0$. This follows from RSW:
    - If $z \in \Omega_\delta$ is at a microscopic distance from A_1, then if $Q_\delta^1(z)$ happens, there is a white path from z to A_2 and to A_3.
    - At least one of A_2 and A_3 is at a macroscopic distance from z. Hence $\mathbb{P}_{\Omega_\delta}\left(Q_\delta^1(z)\right) \to 0$.
  (2) For any $z \in A_1$, $H^2(z) + H^3(z) = 1$. This follows essentially from self-duality:
    - For $z \in \Omega_\delta$, $\mathbb{P}_{\Omega_\delta}\left(Q_\delta^2(z)\right) = \mathbb{P}_{\Omega_\delta}\left(\tilde{Q}_\delta^2(z)\right)$, where $\tilde{Q}_\delta^2 = \{a_2, z \|_\mathrm{b} A_2\}$.
    - At least one of $Q_\delta^3(z)$ and $\tilde{Q}_\delta^2(z)$ happens by self-duality and $Q_\delta^3(z) \cap \tilde{Q}_\delta^2(z) = \emptyset$.
    - This is 'regular' by RSW (i.e. we can exchange limit $z \to A_1$ and $\delta \to 0$).
  (3) As $z \in A_1$ moves from a_2 to a_3, $H^3(z)$ strictly increases from 0 to 1.
    - Let $z,\ \tilde{z} \in A_1$ with z closer to a_2 than $\tilde{z}$. We have that $Q_\delta^3(z) \subset Q_\delta^3(\tilde{z})$, so
$$H_\delta^3(\tilde{z}) - H_\delta^3(z) = \mathbb{P}_{\Omega_\delta}\left[Q_\delta^3(\tilde{z}) \setminus Q_\delta^3(z)\right].$$
      One considers $z, z', z'', \tilde{z}$ in c.c.w order, and two disjoint tubes: 1- T_1 which links $[z, z']$ and A_3 and 2- T_2 which links $[z'', \tilde{z}]$ and A_2. If there exists a black path in T_1 crossing from $[z, z']$ to A_3 and a white path in T_2 crossing from $[z'', \tilde{z}]$ to A_2, then we are in the set of configurations $Q_\delta^3(\tilde{z}) \setminus Q_\delta^3(z)$. Hence since the two tubes are disjoint:
$$H_\delta^3(\tilde{z}) - H_\delta^3(z) \geq \mathbb{P}\left[[z, z'] \leftrightsquigarrow_w A_3 \text{ in } T_1\right] \mathbb{P}\left[[z'', \tilde{z}] \leftrightsquigarrow_w A_2 \text{ in } T_2\right]$$
    - By RSW, the probabilities of two latter events are strictly positive, so $H_\delta^3(\tilde{z}) - H_\delta^3(z)$ is uniformly positive as $\delta \to 0$.
    - By RSW, making concentric annuli, we see that $H_\delta^3(a_3) \to 1$ as $\delta \to 0$ (we can make more and more concetric annuli around a_3 when $\delta \to 0$ and for each annulus, there is a strictly positive probability of seeing a white path separating a_3 from A_3).

1.10. **Discrete Cauchy-Riemann Equations.**
- This is the key identity to prove analyticity (which is itself the key property).
- For an oriented edge $\vec{e} \in \Omega_\delta$ from vertex $x \in \Omega_\delta$ to vertex $y \in \Omega_\delta$, and a function $f : \Omega_\delta \to \mathbb{C}$, we define the discrete derivative $\partial_{\vec{e}} f$ by $f(y) - f(x)$.

1.10.1. *Holomorphic Functions: Cauchy-Riemann Equations and Morera's theorem.* Some reminder on holomorphic functions:

- A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if it satisfies the *Cauchy-Riemann* equations:

$$\begin{aligned} \frac{\partial}{\partial x} \mathfrak{Re} f(z) &= \frac{\partial}{\partial y} \mathfrak{Im} f(z), \\ \frac{\partial}{\partial y} \mathfrak{Re} f(z) &= -\frac{\partial}{\partial x} \mathfrak{Im} f(z). \end{aligned}$$

  We can write it in a "simpler" form which will be usefull for Cardy's formula. We denote $f^{(1)} = \mathfrak{Re} f$ and $f^{(i)} = \mathfrak{Im} f$ (and similarly, $f^{(-1)} = -\mathfrak{Re} f$ and $f^{(-i)} = -\mathfrak{Im} f$). With the definition: $f(z+h) = f(z) + \partial_h f(z) h + o(|h|)$, we see that the Cauchy-Riemann equations can be written as:

$$\partial_1 f^{(1)} = \partial_i f^{(i)}, \quad \partial_i f^{(1)} = \partial_{-1} f^{(i)}$$

  or in a shorter form: for any $\mu, \nu \in \{1, i\}$

$$\partial_\mu f^{(\nu)} = \partial_{i\mu} f^{(i\nu)}. \tag{1.1}$$

- In the next step, i.e. to prove the holomorphicity of φ, we will use *Morera's theorem*: in a simply connected domain, a function is holomorphic if and only if for any closed smooth loop γ, $\oint_\gamma f(z) dz = 0$.
- Why Morera's criterion holds: we can define the antiderivative $F(z) := \int_w^z f(\zeta)\, \mathrm{d}\zeta$ (i.e. the definition is independent of the contour because of the condition on f) and see that $F(z)$ is holomorphic with complex derivative f. The derivative of a holomorphic function is also holomorphic, hence the result.

1.10.2. *Modified discrete Cauchy-Riemann Equations for H_δ^μ.*

- For the functions H_δ^μ, we can write $\partial_{\vec{e}} H_\delta^\mu = \partial_{\vec{e}}^+ H_\delta^\mu - \partial_{\vec{e}}^- H_\delta^\mu$, where $\partial_{\vec{e}}^+ H_\delta^\mu = \mathbb{P}_{\Omega_\delta}(Q_\delta^\mu(y) \setminus Q_\delta^\mu(x))$ and $\partial_{\vec{e}}^- H_\delta^\mu = \mathbb{P}_{\Omega_\delta}(Q_\delta^\mu(x) \setminus Q_\delta^\mu(y))$. If $\vec{e}^{-1}$ is the same edge with reverse orientation, $\partial_{\vec{e}}^- H_\delta^\mu = \partial_{\vec{e}^{-1}}^+ H_\delta^\mu$.
- Set $\tau = e^{2\pi i/3}$, write $\tau\vec{e}$ for the rotation of $\vec{e}$ around its origin x by $2\pi/3$ and set $H_\delta^1, H_\delta^\tau, H_\delta^{\tau^2} := H_\delta^1, H_\delta^2, H_\delta^3$.
- Key Lemma: The modified discrete Cauchy-Riemann equations hold: for $\mu \in \{1, \tau, \tau^2\}$ and an oriented edge $\vec{e}$, we have

$$\partial_{\vec{e}}^+ H_\delta^\mu = \partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu} = \partial_{\tau^2\vec{e}}^+ H_\delta^{\tau^2\mu}.$$

- Remark: This is a modified and discretized version of Equation (1.1).
- Proof of the modified discrete Cauchy-Riemann equations: we suppose $\mu = 1$, and $\vec{e}$ is a horizontal edge from left to right, let $z, w \in \Omega_\delta$ be the target endpoints of $\tau\vec{e}$ and $\tau^2\vec{e}$, and let us prove the first identity (everything is symmetric)
  - We have that $\partial_{\vec{e}}^+ H_\delta^\mu$ is the probability of $Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$: this event means that there is a white path $\gamma : A_2 \leftrightsquigarrow A_3$ passing between x and y and that there is a black path $\lambda : A_1 \leftrightsquigarrow \{x, z, w\}$.
  - We have that $\partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu}$ is the probability of $Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)$: this event means that there is a white path $\tilde{\gamma} : A_1 \leftrightsquigarrow A_3$ passing between x and z a black path $\tilde{\lambda} : A_2 \leftrightsquigarrow \{x, w, y\}$.
  - In order to show that $\partial_{\vec{e}}^+ H_\delta^\mu = \partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu}$, i.e. $\mathbb{P}[Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)] = \mathbb{P}[Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)]$, we construct a bijection between the $\omega \in Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$ and the $\tilde{\omega} \in Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)$: because each configuration has the same probability, this will prove the identity.
  - Bijection between the $\omega \in Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$ and the $\tilde{\omega} \in Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)$:
    * We consider $\omega \in Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$. There exists a white path $\gamma : A_2 \leftrightsquigarrow A_3$ passing between x and y and a black path $A_1 \leftrightsquigarrow \{x, z, w\}$.
    * Let γ_2 be the cw-most white path from $\vec{e}$ to A_2, let λ be the ccw-most black path from $\vec{e}$ to A_1 and let γ_3 be a the part of γ that goes from e to A_3.
    * Flip the color of all the hexagons that on the ccw side of γ_2 and on the cw side of λ (but not the one belonging to γ_2 and λ): the white path γ_3 becomes black, and this map is clearly invertible. (we can invert it by applying the same procedure)
    * Flip the color of all the hexagons: λ and γ_3 become white, γ_2 becomes black and we get a configuration $\tilde{\omega} \in Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)$.
  - Hence, we have constructed a bijection, and $\partial_{\vec{e}}^+ H_\delta^\mu = \partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu}$.

1.11. **Analyticity.**

- To show analyticity, we use Morera's criterion: we will show that $\psi := H_1 + \tau H_\tau + \tau^2 H_{\tau^2}$ and $\sigma := H_1 + H_\tau + H_{\tau^2}$ (notice $\varphi = \frac{1}{3}\sigma + \frac{2}{3}\psi$) are analytic using Morera's criterion, by computing Riemann sums on lattice-level and seeing that they tend to 0. We begin with ψ.
- Let $\gamma \subset \Omega$ be a simple smooth curve oriented in c.c.w direction, and let γ_δ be a honeycomb discretization of γ that uses $\mathcal{O}\left(\delta^{-1}\right)$ edges, oriented in ccw direction. Let us call U_δ the interior of γ_δ.
- Proof of the analyticity:
  (1) *Discretization:* Discretize $\oint_\gamma \psi\left(z\right) \mathrm{d}z$ by $I_\delta(\gamma,\psi) := \sum_{\vec{e}\in U_\delta} \psi_\delta\left(\vec{e}\right)\cdot\vec{e}$, where we write $\psi_\delta\left(\vec{e}\right) := \frac{1}{2}\left(\psi_\delta\left(y\right)+\psi_\delta\left(x\right)\right)$ and $\vec{e} := (y-x)$ (we will identify oriented edges with complex numbers).
  (2) *Transforming contour integration into face integration:* We can write

$$\sum_{\vec{e}\in\gamma_\delta} \psi_\delta\left(\vec{e}\right)\cdot\vec{e} = \sum_{f\in\mathcal{F}_{U_\delta}} \sum_{\vec{e}\in\partial f} \psi_\delta\left(\vec{e}\right)\cdot\vec{e},$$

  where $\mathcal{F}_{U_\delta}$ is the set of hexagonal faces of U_δ and ∂f is the boundary of f, oriented ccw (this is because an edge which is not on the curve will appear twice, and the contributions will have two different signs).
  (3) *Make $\partial_{\vec{e}} H_\delta^\mu$ appears using integration by part*: We can rewrite $\sum_{\vec{e}\in\partial f}\psi_\delta\left(\vec{e}\right)\cdot\vec{e} = -\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta m\left(\vec{e}\right)$ by discrete resummation, where $m\left(\vec{e}\right)$ is the midpoint of $\vec{e}$. Indeed, more generally, if ψ and ϕ are two functions:

$$\begin{aligned}
\sum_{\vec{e}\in\partial f}\psi(\vec{e})\partial_{\vec{e}}\phi &= \frac{1}{2}\sum_{\vec{e}\in\partial f}\left(\psi(y)+\psi(x)\right)\left(\phi(y)-\phi(x)\right)\\
&= \frac{1}{2}\sum_{\vec{e}\in\partial f}\psi(y)\phi(y)-\psi(y)\phi(x)+\psi(x)\phi(y)-\psi(x)\phi(x)\\
&= \frac{1}{2}\sum_{\vec{e}\in\partial f}\psi(x)\phi(x)-\psi(y)\phi(x)+\psi(x)\phi(y)-\psi(y)\phi(y)\\
&= \frac{1}{2}\sum_{\vec{e}\in\partial f}\left(\psi(x)-\psi(y)\right)\phi(x)+\left(\psi(x)-\psi(y)\right)\phi(y)\\
&= \sum_{\vec{e}\in\partial f}\left(\psi(x)-\psi(y)\right)\frac{\left(\phi(x)+\phi(y)\right)}{2} = -\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi\phi(\vec{e}),
\end{aligned}$$

  where we used the notation $\vec{e} = (x,y)$. If we apply this to $\psi = \psi_\delta$ and $\phi(z) = z$, we obtain the equality $\sum_{\vec{e}\in\partial f}\psi_\delta\left(\vec{e}\right)\cdot\vec{e} = -\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta m\left(\vec{e}\right)$.
  (4) *Deal with obvious simplifications:* Each term $\partial_{\vec{e}}\psi_\delta m\left(\vec{e}\right)$ can be big: yet since $\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta = 0$, we have $\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta m\left(\vec{e}\right) = \sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta\left[m\left(\vec{e}\right)-\alpha_f\right]$ for any complex α_f. If we consider $\alpha_f = c(f)$ the center of the face f, using the fact that $m(\vec{e}) - c(f) = \frac{1}{2}\vec{e}^*$ where $\vec{e}^*$ is the oriented edge of the dual of Ω_δ that crosses $\vec{e}$ oriented such that $\vec{e}^*/\left(i\vec{e}\right) > 0$, we get $\sum_{\vec{e}\in\partial f}\psi_\delta\left(\vec{e}\right)\cdot\vec{e} = -\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta m\left(\vec{e}\right) = \frac{1}{2}\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta\cdot\vec{e}^*$.
  (5) *Summing over the faces:* With the sum over the faces, we have to study

$$I_\delta(\gamma,\psi) = \frac{1}{2}\sum_{f\in\mathcal{F}_{U_\delta}}\sum_{\vec{e}\in\partial f}\partial_{\vec{e}}\psi_\delta\cdot\vec{e}^*.$$

  Consider a given orientation of the edges, denoted by $\vec{\mathcal{E}}_{U_\delta}$ (i.e. we choose one orientation for each non-oriented edge): an edge $\vec{e}\in\vec{\mathcal{E}}_{U_\delta}$ which is inside U_δ and not on γ_δ appears once in the sum $I_\delta(\gamma,\psi)$ and its inverse $\vec{e}^{-1}$ also, besides, since $\partial_{\vec{e}^{-1}}\psi = -\partial_{\vec{e}}\psi$ and $(\vec{e}^{-1})^* = -\vec{e}^*$, the two contributions are equal. Thus, resumming over all the edges $\vec{e}\in\vec{\mathcal{E}}_{U_\delta}$, we get $I_\delta\left(\gamma,\psi\right) = \sum_{\vec{e}\in\vec{\mathcal{E}}_{U_\delta}}\vec{e}^*\partial_{\vec{e}}\psi_\delta$ + boundary terms. The boundary terms tend to 0 as $\delta\to 0$: there are $\mathcal{O}\left(\frac{1}{\delta}\right)$ of them, and they are of order $o\left(\delta\right)$ (the δ comes from the edge length, the $o\left(1\right)$ from $\partial_{\vec{e}}\psi$, by RSW).
  (6) *Definition of ψ, making $\partial_{\vec{e}}^- H_\delta^\mu$ disappears:* $\psi_\delta = H_\delta^1 + \tau H_\delta^\tau + \tau^2 H_\delta^{\tau^2}$, hence

$$\partial_{\vec{e}}\psi_\delta = \partial_{\vec{e}}^+ H_\delta^1 + \tau\partial_{\vec{e}}^+ H_\delta^\tau + \tau^2\partial_{\vec{e}}^+ H_\delta^{\tau^2} - \partial_{\vec{e}^{-1}}^+ H_\delta^1 - \tau\partial_{\vec{e}^{-1}}^+ H_\delta^\tau - \tau^2\partial_{\vec{e}^{-1}}^+ H_\delta^{\tau^2}$$

and thus $\sum_{\vec{e}\in\vec{\mathcal{E}}_{U_\delta}} \vec{e}^* \partial_{\vec{e}}\psi_\delta$ is equal to:

$$\sum_{\vec{e}\in\vec{\mathcal{E}}_{U_\delta}} \vec{e}^* \left[\partial^+_{\vec{e}} H^1_\delta + \tau\partial^+_{\vec{e}} H^\tau_\delta + \tau^2\partial^+_{\vec{e}} H^{\tau^2}_\delta\right] + \sum_{\vec{e}\in\vec{\mathcal{E}}_{U_\delta}} (-\vec{e}^*) \left[\partial^+_{\vec{e}^{-1}} H^1_\delta + \tau\partial^+_{\vec{e}^{-1}} H^\tau_\delta + \tau^2\partial^+_{\vec{e}^{-1}} H^{\tau^2}_\delta\right]$$

and since $(\vec{e}^{-1})^* = -\vec{e}^*$, the second term is $\sum_{\vec{e}\in\vec{\mathcal{E}}_{U_\delta}} \left(\vec{e}^{-1}\right)^* \left[\partial^+_{\vec{e}^{-1}} H^1_\delta + \tau\partial^+_{\vec{e}^{-1}} H^\tau_\delta + \tau^2\partial^+_{\vec{e}^{-1}} H^{\tau^2}_\delta\right]$ and thus,

$$I_\delta(\gamma,\psi) \sim \sum_{\vec{e}\in\vec{\mathcal{E}}_{U_\delta}} \vec{e}^*\partial_{\vec{e}}\psi_\delta = \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^* \left[\partial^+_{\vec{e}} H^1_\delta + \tau\partial^+_{\vec{e}} H^\tau_\delta + \tau^2\partial^+_{\vec{e}} H^{\tau^2}_\delta\right]$$

where on the r.h.s. $\mathcal{E}_{U_\delta}$ is the set of all oriented edges (note that $\vec{\mathcal{E}}_{U_\delta}$ was a choice of orientation, i.e. for each unoriented edge $e = \{x, y\}$ either $\overrightarrow{xy}$ or $\overrightarrow{yx}$ was in $\vec{\mathcal{E}}_{U_\delta}$, whereas both $\overrightarrow{xy}$ and $\overrightarrow{yx}$ are in $\mathcal{E}_{U_\delta}$).

(7) *Making $\partial^+_{\tau\vec{e}} H^{\tau\mu}_\delta$ and $\partial^+_{\tau^2\vec{e}} H^{\tau^2\mu}_\delta$ appear:* We split the sum:

$$I_\delta(\gamma,\psi) \sim \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^*\partial^+_{\vec{e}} H^1_\delta + \tau \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^*\partial^+_{\vec{e}} H^\tau_\delta + \tau^2 \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^*\partial^+_{\vec{e}} H^{\tau^2}_\delta$$

and we do a resummation for the second and third terms: up to boundary terms which will disappear, we have

$$I_\delta(\gamma,\psi) \sim \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^*\partial^+_{\vec{e}} H^1_\delta + \tau \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} (\tau\vec{e})^*\, \partial^+_{\tau\vec{e}} H^\tau_\delta + \tau^2 \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \left(\tau^2\vec{e}\right)^*\partial^+_{\tau^2\vec{e}} H^{\tau^2}_\delta$$

and thus

$$I_\delta(\gamma,\psi) \sim \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^*\partial^+_{\vec{e}} H^1_\delta + \tau^2 \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} (\vec{e})^*\, \partial^+_{\tau\vec{e}} H^\tau_\delta + \tau^4 \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} (\vec{e})^*\, \partial^+_{\tau^2\vec{e}} H^{\tau^2}_\delta$$

(8) *Cauchy Riemann Equation:* we use the fact that $\partial^+_{\vec{e}} H^1_\delta = \partial^+_{\tau\vec{e}} H^\tau_\delta = \partial^+_{\tau^2\vec{e}} H^{\tau^2}_\delta$, hence

$$I_\delta(\gamma,\psi) \sim \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^*\partial^+_{\vec{e}} H^1_\delta + \tau^2 \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} (\vec{e})^*\, \partial^+_{\vec{e}} H^1_\delta + \tau^4 \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} (\vec{e})^*\, \partial^+_{\vec{e}} H^1_\delta$$

hence using $\tau^4 = \tau$,

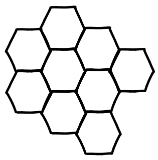
$$I_\delta(\gamma,\psi) \sim (1+\tau+\tau^2) \sum_{\vec{e}\in\mathcal{E}_{U_\delta}} \vec{e}^*\partial^+_{\vec{e}} H^1_\delta$$

(9) *τ is a root of unity:* since $\tau^3 - 1 = 0$, we get that $1 + \tau + \tau^2 = 0$ and thus $I_\delta(\gamma,\psi) \sim 0$, i.e. $\lim_{\delta\to 0} I_\delta(\gamma,\psi) = 0$, and hence ψ is analytic.

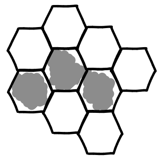
(10) Similarly, for $\sigma := H_1 + H_\tau + H_{\tau^2}$, we get a cancelation because $1 + \tau + \tau^2 = 0$ and σ is analytic as well. Hence φ is holomorphic.

Percolation (Figures)

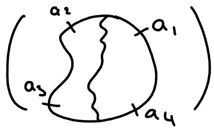
Honeycomb lattice



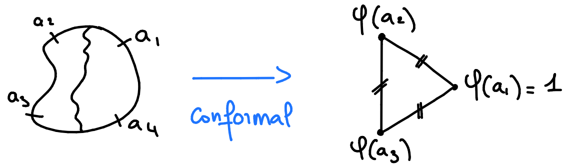
Face percolation



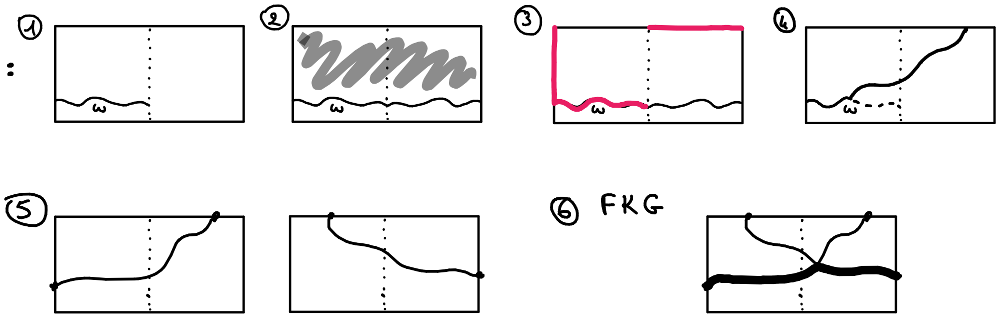
Cardy's formula : $\mathbb{P}^{\delta}_{1/2}$



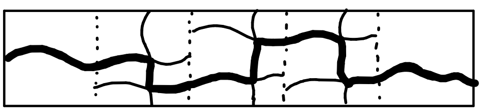
$\xrightarrow[\delta \to 0]{}$ $\mathrm{Re}\ \varphi(a_4)$



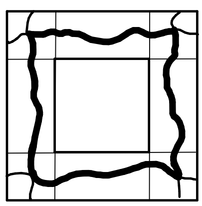
RSW :



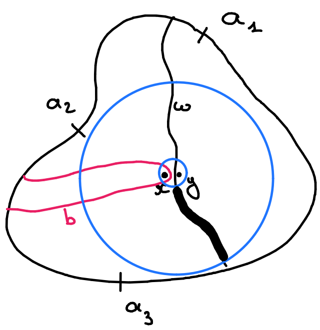
RSW :

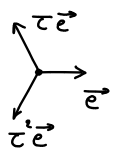


Annulus estimate

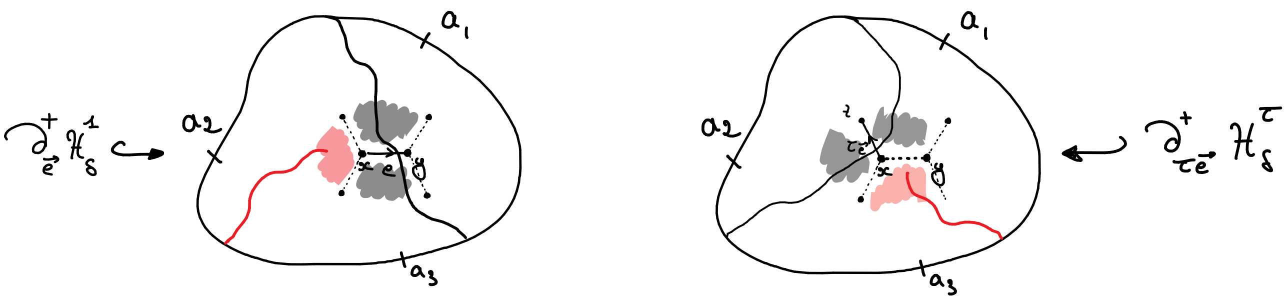


Hölder Condition:

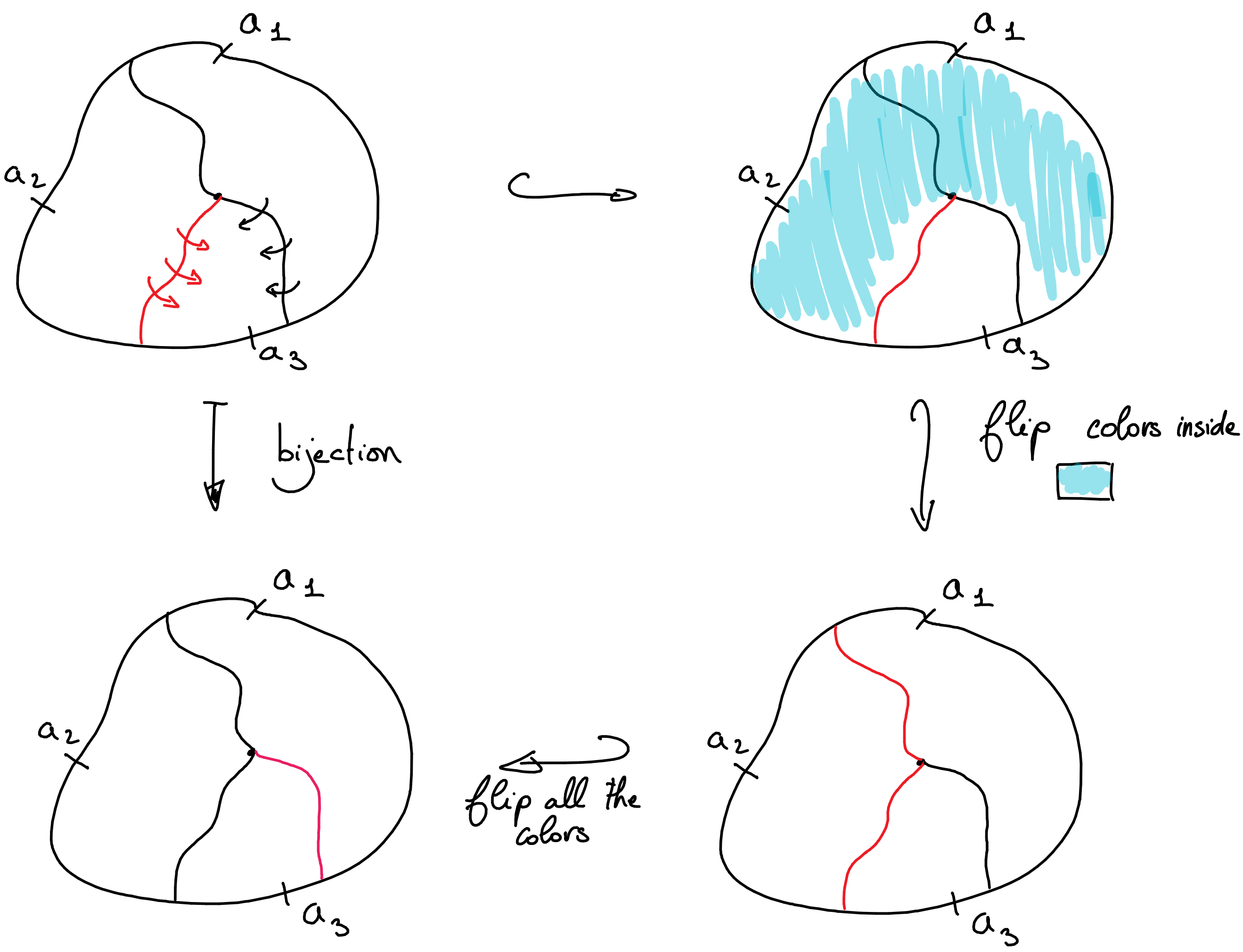




Cauchy Riemann equations:



Bijection:



LATTICE MODELS (MA1, EPFL)

1. Ising Model

1.1. The Ising Model: Definition, Magnetization and Phase transition:

1.1.1. *Definition.*

- We consider a finite graph $\mathbb{G}$, with vertices $\mathbb{V}$ and edges $\mathbb{E}$, we consider a subset $\partial\mathbb{G} \subset \mathbb{V}$ and we assume that two vertices of $\partial\mathbb{G}$ are not connected by an edge.
- The Ising model is a random assignment of ± 1 spins on $\mathbb{V}_{\text{int}} := \mathbb{V} \setminus \partial\mathbb{G}$. On the boundary $\partial\mathbb{G}$ the spins will be fixed: it is the boundary condition. If $b : \partial\mathbb{G} \to \pm 1$ is a boundary condition, the Ising model is a probability on
$$\{+1,-1\}_b^{\mathbb{V}} := \left\{\sigma \in \{+1,-1\}^{\mathbb{V}} \mid \sigma(x) = b(x), \forall x \in \partial\mathbb{G}\right\}.$$
- The main boundary conditions that we will consider are:
  - the $+$ boundary condition, when $\partial\mathbb{G} \neq \emptyset$ and $b(x) = +1$ for $x \in \partial\mathbb{G}$,
  - the $-$ boundary condition, when $\partial\mathbb{G} \neq \emptyset$ and $b(x) = -1$ for $x \in \partial\mathbb{G}$,
  - the free boundary condition, when $\partial\mathbb{G} = \emptyset$.
- The probability of a spin configuration $(\sigma_x)_{x\in\mathbb{V}}$ is proportional to $e^{-\beta H(\sigma)}$, where $H(\sigma) = -\sum_{\langle ij\rangle\in\mathbb{E}} \sigma_i\sigma_j$ is the energy (the sum is over all unoriented edges) and $\beta > 0$ is the inverse temperature.
- In other words, in the Ising model with boundary condition b, and inverse temperature β, for any $\sigma \in \{+1,-1\}_b^{\mathbb{V}}$,
$$\mathbb{P}_{b,\beta}^{\mathbb{G}}\{\sigma\} = e^{-\beta H(\sigma)}/Z_\beta,$$
where $Z_\beta = \sum_{\tilde\sigma\in\{+1,-1\}_b^{\mathbb{V}}} e^{-\beta H(\tilde\sigma)}$ is the partition function. We will often omit one or more of the indices $\mathbb{G}, \beta, b$ when it can be deduced from the context.
- When $\beta \to 0$, the measure converges to the uniform measure on the configuration space: a lot of disagreements between spins, a lot of disorder, this represents a system at high temperature. When $\beta \to \infty$, the spin configuration "freezes", the measure converge to a uniform measure on the minima of the energy H (for ex., in the $+1$ boundary condition, it converges to the Dirac measure on the constant configuration $\sigma_x = +1$): this represents a system at low temperature. This explains why β is called the inverse temperature.
- Remark: we can generalize the Ising model by introducing an external magnetic field (add $h\sum_i \sigma_i$ to $\beta H(\sigma)$), replacing $\sigma_i\sigma_j$ by general couplings $J_{ij}\sigma_i\sigma_j$, ..., but we will not do that (not that it is not interesting, but things become more difficult...)

1.1.2. *Global flip symmetry.*

- If $\sigma = (\sigma_x)_{x\in\mathbb{V}}$ is a random configuration with law $\mathbb{P}_b^{\mathbb{G}}$, then $-\sigma = (-\sigma_x)_{x\in\mathbb{V}}$ is a random configuration with law $\mathbb{P}_{-b}^{\mathbb{G}}$. Indeed, $\mathbb{P}(-\sigma = s) = \mathbb{P}(\sigma = -s) \sim e^{-\beta H(-s)} = e^{-\beta H(s)}$ and the only thing which changed is that $-\sigma \in \{-1,1\}_{-b}^{\mathbb{V}}$.
- In particular, if σ follows the law of the Ising model with free boundary condition, then $-\sigma$ has the same law. This implies that for any vertices $x_1, \dots, x_n$
$$\mathbb{E}_{\text{free}}^{\mathbb{G}}\left[\prod_{i=1}^n \sigma_{x_i}\right] = 0$$
if n is odd since $\mathbb{E}_{\text{free}}^{\mathbb{G}}\left[\prod_{i=1}^n \sigma_{x_i}\right] = \mathbb{E}_{\text{free}}^{\mathbb{G}}\left[\prod_{i=1}^n -\sigma_{x_i}\right] = (-1)^n\mathbb{E}_{\text{free}}^{\mathbb{G}}\left[\prod_{i=1}^n \sigma_{x_i}\right].$

1.1.3. *Motivation.*

- The Ising model was introduced as a model of ferromagnetism: one would like to study how small magnets that tend to align locally (in ferromagnetic material) behave, and to understand why at low temperature, they tend to align globaly whereas they don't at high temperature.
- Explaining clearly why this happens for a number of materials (such as iron) requires quantum mechanics and is difficult.

- Anyway, the model is a strong simplification of reality (spins only take two directions, we don't have magnetic field and this is not a quantum model).
- Nevertheless, there is a strong support for universality.
- General postulate in statistical mechanics: the probability of a configuration (in the so-called canonical ensemble) is proportional to its Boltzmann weight $\exp(-\beta H)$. In fact, this is the family of probability measures obtained by maximizing the entropy:
$$\mathrm{Ent}(\mu) = -\sum_{\sigma} \mu(\sigma)\log(\mu(\sigma))$$
in the space of probability measures with fixed mean energy (i.e. $\sum_\sigma \mu(\sigma)H(\sigma) = \alpha$).

1.1.4. *Influence of the boundary condition on the boundary and phase transition.*

- We consider a square-grid discretization Ω_δ with mesh δ of a planar domain such that $0 \in \Omega$, and with boundary $\partial\Omega_\delta$.
- When $\beta \to \infty$, the system freezes, hence the boundary has a large influence on the spin at 0. When $\beta \to 0$, the spins inside the domain become independent: the boundary has no influence on the spin at 0.
- How do the boundary condition influences the spin at 0 (and more generally the measure) when there are more and more spins (i.e. when the mesh δ goes to 0) ?
- General Results on the Magnetisation: We consider the Ising model on Ω_δ. The one with + boundary condition is denoted by $\mathbb{E}_\delta^+$. We will show that
  - For any boundary condition η (which can be the free boundary condition),
  $$\mathbb{E}_\delta^-[\sigma_0] \leq \mathbb{E}_\delta^\eta[\sigma_0] \leq \mathbb{E}_\delta^+[\sigma_0].$$
  (i.e. the + boundary condition is the one which has the most "positive" influence on the spin at 0.)
  - $\delta \to \mathbb{E}_\delta^+[\sigma_0]$ is decreasing in δ: it converges as $\delta \to 0$ to the magnetization denoted by $\langle \sigma_0 \rangle_\beta^+$. (i.e. the more the boundary is far away, the less it has an influence on 0)
  - $\beta \to \langle \sigma_0 \rangle_\beta^+$ is:
    * increasing in β (when the temperature decreases, the system tends to freeze and the boundary condition has more influence),
    * equal to 0 when $\beta \ll 1$ (when the temperature is too hight, when there is an infinite number of atoms, the boundary condition does not influence the spin in the bulk),
    * strictly positive when $\beta \gg 1$,
    * thus, there is a phase transition between the βs such that $\langle \sigma_0 \rangle_\beta^+ = 0$ and the ones such that $\langle \sigma_0 \rangle_\beta^+ > 0$.
  - We consider $0 < \beta_c = \sup\left\{\beta, \langle \sigma_0 \rangle_\beta^+ = 0\right\} < \infty$. Using a duality argument, the Kramers-Wannier duality, we will show that if the critical β is the self-dual point, then $\beta_c = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$.

1.2. **Boundary conditions and conditionning.**

1.2.1. *Relation between the + and the free boundary conditions.*

- We can consider the graph $\mathbb{G}^{\mathrm{g}}$ obtained from $\mathbb{G}$ by merging the boundary vertices $\partial\mathbb{G}$ into one vertex g (which is the new boundary). The restriction on $\{+1,-1\}^{\mathbb{V}_{\mathrm{int}}}$ of the + boundary Ising measure on $\mathbb{G}$ is equal to the restriction on $\{+1,-1\}^{\mathbb{V}_{\mathrm{int}}}$ of the + boundary Ising measure on $\mathbb{G}^{\mathrm{g}}$.
- The + boundary and free boundary Ising measures on $\mathbb{G}^g$ are related. If one considers the free boundary Ising measure on $\mathbb{G}^{\mathrm{g}}$ and conditions on the fact that $\sigma_g = +1$, one obtains the Ising measure on $\mathbb{G}^{\mathrm{g}}$ with + boundary condition.
- Thus, for any $A \subset \mathbb{V}_{\mathrm{int}}$, if we denote $\sigma_A := \prod_{x\in A}\sigma_x$, we get
$$\mathbb{E}_+^{\mathbb{G}^{\mathrm{g}}}[\sigma_A] = \mathbb{E}_f^{\mathbb{G}^{\mathrm{g}}}[\sigma_A \mid \sigma_{\mathrm{g}} = +1] = \frac{\mathbb{E}_f^{\mathbb{G}^{\mathrm{g}}}\left[\sigma_A 1_{\sigma_{\mathrm{g}}=+1}\right]}{\mathbb{P}_f^{\mathbb{G}^{\mathrm{g}}}[\sigma_{\mathrm{g}} = +1]}$$
and using that $1_{\sigma_{\mathrm{g}}=1} = \frac{1}{2}(\sigma_{\mathrm{g}}+1)$, and that, from the spin flip symmetry, $\mathbb{P}_f^{\mathbb{G}^{\mathrm{g}}}[\sigma_{\mathrm{g}} = +1] = \frac{1}{2}$,
$$\mathbb{E}_+^{\mathbb{G}^{\mathrm{g}}}[\sigma_A] = \mathbb{E}_f^{\mathbb{G}^{\mathrm{g}}}[\sigma_A\sigma_{\mathrm{g}}] + \mathbb{E}_f^{\mathbb{G}^{\mathrm{g}}}[\sigma_A].$$

Using again the spin flip symmetry, we get that

$$\mathbb{E}_{+}^{\mathbb{G}}[\sigma_A] = \mathbb{E}_{+}^{\mathbb{G}^{\mathrm{g}}}[\sigma_A] = \begin{cases} \mathbb{E}_{f}^{\mathbb{G}^{\mathrm{g}}}[\sigma_A \sigma_{\mathrm{g}}] & \text{if } |A| \text{ is odd,} \\ \mathbb{E}_{f}^{\mathbb{G}^{\mathrm{g}}}[\sigma_A] & \text{if } |A| \text{ is even.} \end{cases}$$

1.2.2. *Markov Property.*

- Let $b : \partial\mathbb{G} \to \{+1, -1\}$ be a boundary condition. We consider an induced subgraph $\mathcal{H} \subset \mathbb{G}$ (if $x, y \in \mathcal{H}$ and $x \sim y$ in $\mathbb{G}$ then the edge $\{x, y\}$ is an edge in $\mathcal{H}$), $\partial_{\mathbb{G}}\mathcal{H} = \{x \in \mathbb{G} \setminus \mathcal{H}, \exists y \in \mathcal{H}, x \sim y\} \subset \mathbb{V}_{\text{int}}$, $w : \mathbb{G} \setminus \mathcal{H} \to \{+1, -1\}$ such that $w_{|\partial\mathbb{G}} = b$ and σ a random configuration coming from the Ising model on $\mathbb{G}$ with boundary condition b.
- The law of $\sigma_{|\mathcal{H}}$ conditionned on the fact that $\sigma_{|\mathbb{G}\setminus\mathcal{H}} = w$ is the Ising law on $\mathcal{H}$ with boundary condition $w_{|\partial\mathcal{H}}$. Indeed,

$$\mathbb{P}\left[\sigma_{|\mathcal{H}} = s \mid \sigma_{|\mathbb{G}\setminus\mathcal{H}} = w\right] = \frac{\mathbb{P}\left[\left(\sigma_{|\mathcal{H}} = s\right) \& \left(\sigma_{|\mathbb{G}\setminus\mathcal{H}} = w\right)\right]}{\mathbb{P}\left[\sigma_{|\mathbb{G}\setminus\mathcal{H}} = w\right]} \propto \mathbb{P}\left[\left(\sigma_{|\mathcal{H}} = s\right) \& \left(\sigma_{|\mathbb{G}\setminus\mathcal{H}} = w\right)\right] \propto e^{-\beta H(s)}$$

where $H(s) = -\sum_{x\sim y, x\in\mathcal{H}, y\in\mathcal{H}} s_x s_y - \sum_{x\sim y, x\in\mathcal{H}, y\in\partial\mathcal{H}} s_x w_y$.
- Remark : We will use this to relate the Ising model with + boundary condition on D with the Ising model with + boundary condition on $D' \supset D$: the first can be obtained from the second by conditionning all the spins outside D to be equal to $+1$.
- Markov Property of the Ising Model: if we condition an Ising model to take some values on a subset Λ such that $\mathbb{G} \setminus \Lambda$ has two connected components (e.g. Λ is a closed curve in the graph with an inside and outside), then what we see in each components are two independent Ising models (with boundary conditions which can be deduced from the one of the initial model and the values on Λ we conditionned on).

1.3. **Simulation.**

1.3.1. *Algorithms.*

- **Metropolis algorithm/Glauber dynamics**: start from an arbitrary configuration (with the boundary values as specified by the Ising measure we want to sample), and make random flips:
  - Compute the energy of the current configuration H_σ.
  - Pick a vertex x at random (in $\mathbb{V}_{\text{int}}$), consider the configuration ρ, obtained by flipping the spin x of σ, and compute its energy H_ρ.
  - If $H_\rho \leq H_\sigma$, replace σ by ρ. If $H_\rho > H_\sigma$, replace σ by ρ with probability $e^{-\beta H_\rho}/e^{-\beta H_\sigma}$ (i.e. the relative probabilities of ρ and σ).
  - This defines a Markov chain on the state space $\mathcal{S} := \{\pm 1\}_{\text{b}}^{G}$, with transition matrix $P_M = (P_M)_{\rho\sigma}$.
- **Heat bath dynamics:** start from an arbitrary configuration (with the boundary values as specified by the Ising measure we want to sample), and make random flips
  - Compute the energy of the current configuration.
  - Pick a vertex x at random, and sample the spin σ_x at random by giving probability

$$\mathbb{P}\{\sigma_x\} = \frac{e^{-\beta H[\sigma^+]}}{e^{-\beta H[\sigma^+]} + e^{-\beta H[\sigma^-]}},$$

  where σ^+ and σ^- denote the configuration σ, with the spin σ_x forced to be $+1$ and -1 respectively.
  - This define a Markov chain on $\mathcal{S}$, with transition matrix P_H.
- Theorem: for any initial probability measure μ on $\mathcal{S}$, we have that $\mu P_M^n \to \mu_{\text{Ising}}^{\beta}$ and $\mu P_H^n \to \mu_{\text{Ising}}^{\beta}$ as $n \to \infty$.
- In order to prove this, we recall some results on Markov Chains.

1.3.2. *Markov Chain and convergence of the measure.*

- Recall the following Markov chain results: if P is the transition matrix of an irreducible aperiodic Markov chain on $\mathcal{S}$ (i.e. there exists N such that for any $m \geq N$, $(P^m)_{\sigma\rho} > 0$ for all $\sigma, \rho \in \mathcal{S}$), then $\mu P^n \to \mu_{\text{stat}}$ as $n \to \infty$, where μ_{stat} is the unique stationary measure, i.e. $\mu_{\text{stat}} P = \mu_{\text{stat}}$. Idea of the proof:
  - We know 1 is an eigenvalue of P, and hence of P^T; we know the eigenvalues of P are ≤ 1 in modulus, so are the ones of P^T.
  - Perron-Frobenius theorem: let Q be a matrix with positive entries. Then the largest eigenvalue (in modulus) is real, simple and the corresponding eigenvector can be taken with positive entries.

  - Apply Perron-Frobenius to $Q = P^N$, and deduce that there is a unique positive eigenvector of eigenvalue 1 for Q (up to multiplication by a positive number), but then it implies the same results for P^T (otherwise, there would be a contradiction).
  - In particular, there exists a unique stationary measure μ_{stat} for P.
  - This implies that μP^n converges to a multiple of the unique stationary measure μ_{stat}. Since μP^n and μ_{stat} are two probability measures, μP^n converges to μ_{stat}.
- Sufficient (but non-necessary criterion) for measure invariance: the detailed balance.
  - A measure μ is stationary if and only if $\sum_\sigma \mu(\sigma)P(\sigma,\rho) = \mu(\rho)$. Using the fact that $\sum_\sigma P(\rho,\sigma) = 1$, we can write the previous equation as

$$\sum_\sigma \mu(\sigma)P(\sigma,\rho) = \sum_\sigma P(\rho,\sigma)\mu(\rho)$$

  - A measure μ such that $\mu(\sigma)P(\sigma,\rho) = \mu(\rho)P(\rho,\sigma)$ for all $\sigma,\rho \in \mathcal{S}$ is said to satisfy the *detailed balance equation*: the "flux" of probability from σ to ρ equals the "flux" of probability of ρ to σ. Using the first point, we obtain the following proposition:
  - Proposition: if μ satisfies the detailed balance, then μ is a stationary measure for P.

1.3.3. *Proof of convergence for the Metropolis and Heat bath dynamics:*

- The Heat Bath and Metropolis dynamics are irreducible and aperiodic:
  - **Heat-bath dynamics:** for any σ,ρ, we consider a path $\sigma = \sigma_0 \to ... \to \sigma_n = \rho$ where σ_{i+1} and σ_i only differ at one vertex, $\mathbb{P}[X_n = \sigma_n, \dots, X_1 = \sigma_1 \mid X_0 = \sigma_0] > 0$ and thus $P_M^n(\sigma,\rho) > 0$. Since in this dynamics, there is a positive probability to stay at the same configuration after one step, we get that for any $m \geq n$, $P_H^m(\sigma,\rho) > 0$. Since there is a finite number of configurations, there exists N such that for any $m \geq N$, for any σ,ρ, $P_H^m(\sigma,\rho) > 0$: the dynamics is irreducible and aperiodic.
  - **Metropolis dynamics:** One has to be careful about something: if the configuration maximises the energy, the probability to stay at this configuration is 0. Using the same arguments as before, for any σ,ρ such that one of the two configurations does not reach the maximal energy, there exists n such that that for any $m \geq n$, $P_M^m(\sigma,\rho) > 0$. It remains to prove this when the two configurations have the maximal energy. In this case, one considers σ^x obtained by flipping the spin at x in the spin configuration σ. If σ^x has maximal energy, then the probability to stay at σ is positive and we can apply the same arguments as before. If the energy of σ^x is smaller than the one of σ, the probability to stay at σ^x is positive: we consider n such that $P_H^n(\sigma_x,\rho) > 0$, then for any m, $P_H^{n+m+1}(\sigma,\rho) \geq P_H(\sigma,\sigma_x)P_H^m(\sigma_x,\sigma_x)P_H^n(\sigma_x,\sigma_\rho) > 0$. This allows us to conclude that the dynamics is irreducible and aperiodic.
- The Ising Measure with inverse temperature β is the unique stationary measure since it satisfies the detailed balance equation.
  - **Metropolis dynamics:** assume $H_\rho > H_\rho$ (the other situation is symmetric) , we have a flux $\frac{1}{|V|}\left(e^{-\beta H_\rho}/e^{-\beta H_\sigma}\right)e^{-\beta H_\sigma} = e^{-\beta H_\rho}$ from σ to ρ, and a flux $\frac{1}{|V|}1 \cdot e^{-\beta H_\rho}$ from ρ to σ, so both are equal – the $\frac{1}{|V|}$ is there because we pick the vertices uniformly.
  - **Heat-bath dynamics:** if σ^-,σ^+ are configurations coinciding except at a vertex x, $\sigma_x^- = -1$ and $\sigma_x^+ = 1$, we have a flux $\frac{1}{|V|}\frac{e^{-\beta H[\sigma^-]}e^{-\beta H[\sigma^+]}}{e^{-\beta H[\sigma^+]}+e^{-\beta H[\sigma^-]}}$ from σ^- to σ^+ and flux $\frac{1}{|V|}\frac{e^{-\beta H[\sigma^+]}e^{-\beta H[\sigma^-]}}{e^{-\beta H[\sigma^+]}+e^{-\beta H[\sigma^-]}}$ from σ^+ to σ^-, which are both equal.
- The convergence of the algorithms is a consequence of the main theorem of the previous subsection.

1.4. **Graphical expansions.**

- We are going to express the partition functions

$$Z = \sum_{\sigma \in \{+1,-1\}_b^{\mathbb{V}}} e^{-\beta H(\sigma)}, \quad Z_A = \sum_{\sigma \in \{+1,-1\}_b^{\mathbb{V}}} \sigma_A e^{-\beta H(\sigma)}$$

where $\sigma_A = \prod_{x \in A} \sigma_x$ using more geometric objects.
- The correlations can be expressed using the fact that $\mathbb{E}_b^{\mathbb{G}}[\sigma_A] = \frac{Z_A}{Z}$.

1.4.1. *General strategy:*

- We write
$$Z_A = \sum_{\sigma\in\{+1,-1\}_b^{\mathbb{V}}} e^{-\beta H(\sigma)}\sigma_A = \sum_{\sigma\in\{+1,-1\}_b^{\mathbb{V}}} \prod_{e\in E} e^{\beta\sigma(x)\sigma(y)}\sigma_A = \sum_{\sigma\in\{+1,-1\}_b^{\mathbb{V}}} \prod_{e\in E} f(\sigma_x\sigma_y)\sigma_A.$$
- The function $f : \sigma \to e^{-\beta\sigma}$ can be decomposed in a basis or family of functions which spans the space of functions on $\{+1, -1\}$. Depending on the family, we get different graphical expansions of the Ising model:
  - **Low Temperature Expansion:** $f_0 : \sigma \to \delta_{\sigma=1}$ and $f_1 : \sigma \to \delta_{\sigma=-1}$:
$$f(\sigma) = e^{\beta}\left(\delta_{\sigma=1} + e^{-2\beta}\delta_{\sigma=-1}\right).$$
  - **High Temperature Expansion:** $f_0 : \sigma \to 1$ and $f_1 : \sigma \to \sigma$: using the fact that $\sigma^{2n} = 1$, $\sigma^{2n+1} = \sigma$, and the Taylor expansion $f(\sigma) = \sum_{n\in\mathbb{N}} \frac{\beta^n}{n!}\sigma^n$, we get
$$f(\sigma) = \sum_{n\in\mathbb{N}} \frac{\beta^{2n}}{(2n)!} + \sum_{n\in\mathbb{N}} \frac{\beta^{2n+1}}{(2n+1)!}\sigma = \cosh(\beta)\left[1 + \tanh(\beta)\sigma\right].$$
  - **F-K Expansion:** $f_0 : \sigma \to 1$ and $f_1 : \sigma \to \delta_{\sigma=1}$:
$$f(\sigma) = e^{\beta}\left(e^{-2\beta} + (1 - e^{-2\beta})\delta_{\sigma=1}\right).$$
  - **Current Expansion:** $(f_i : \sigma \to \sigma^i)_{i\geq 0}$,
$$f(\sigma) = \sum_{n\in\mathbb{N}} \frac{\beta^n}{n!}\sigma^n.$$
- Once we have chosen the family, we decompose: $f = \sum_{i\in I} a_i f_i$, thus
$$Z_A = \sum_{\sigma\in\{+1,-1\}_b^{\mathbb{V}}} \prod_{e\in E}\sum_{i\in I} a_i f_i(\sigma_x\sigma_y)\sigma_A.$$
- We expand the product $\prod_{e\in E}$:
$$Z_A = \sum_{\sigma\in\{+1,-1\}_b^{\mathbb{V}}} \sum_{(i_e)_{e\in E}} \left(\prod_{e\in E} a_{i_e}\right) \prod_{(x,y)=e\in E} f_{i_e}(\sigma_x\sigma_y)\sigma_A$$
  and we can exchange the two summations:
$$Z_A = \sum_{(i_e)_{e\in E}} \left(\prod_{e\in E} a_{i_e}\right) \sum_{\sigma\in\{+1,-1\}_b^{\mathbb{V}}} \prod_{(x,y)=e\in E} f_{i_e}(\sigma_x\sigma_y)\sigma_A.$$
  It remains to compute
$$C(A, (i_e)_{e\in E}) := \sum_{\sigma\in\{+1,-1\}_b^{\mathbb{V}}} \prod_{(x,y)=e\in E} f_{i_e}(\sigma_x\sigma_y)\sigma_A$$
  which will depend on the family $(f_i)_{i\in I}$ we have chosen.

1.4.2. *Low-Temperature Expansion.*
- We consider the + boundary conditions.
- We use the fact that $f(\sigma) = e^{\beta}\left(\delta_{\sigma=1} + e^{-2\beta}\delta_{\sigma=-1}\right)$: from the general strategy (Section 1.4.1), with $f_0(\sigma) = \delta_{\sigma=1}$ and $f_1(\sigma) = \delta_{\sigma=-1}$, we get
$$Z = e^{\beta\#\mathbb{E}} \sum_{\mathcal{E}\subset\mathbb{E}} e^{-2\beta\#\mathcal{E}} \sum_{\sigma\in\{+1,-1\}_+^{\mathbb{V}}} \prod_{(x,y)=e\in\mathcal{E}} \delta_{\sigma_x\neq\sigma_y}$$
- For any $\mathcal{E} \subset \mathbb{E}$, $\sum_{\sigma\in\{+1,-1\}_+^{\mathbb{V}}} \prod_{(x,y)=e\in\mathcal{E}} \delta_{\sigma_x\neq\sigma_y}$ is either equal to 0 or to 1.
  - It is easier to consider the dual $\mathcal{E}^* \subset \mathbb{E}^*$, the condition $\prod_{(x,y)=e\in\mathcal{E}} \delta_{\sigma_x\neq\sigma_y}$ tells us that σ_x and σ_y should be distinct if (x, y) crosses $e^* \in \mathcal{E}^*$.
  - Hence if $\prod_{(x,y)=e\in\mathcal{E}} \delta_{\sigma_x\neq\sigma_y} \neq 0$ then $\mathcal{E}^*$ is the set of disorder edges for σ (i.e. the set of dual edges which separates spins with different values): this set yields a configuration of loops.
  - Besides, since we are considering the + boundary condition, the map $\sigma \to \{e \in \mathbb{E}^*, e \text{ is a disorder edge}\}$ is a bijection between configurations of spins σ and the sets $\mathcal{E}^* \subset \mathbb{E}^*$ which are disjoint union of loops (i.e. every dual vertex is contained in an even number of dual edgers in $\mathcal{E}^*$).

- Hence
$$Z = e^{\beta \#\mathbb{E}} \sum_{\mathcal{E}^* \in \mathcal{C}(\mathbb{E}^*)} e^{-2\beta \#\mathcal{E}^*}$$
where $\mathcal{C}(\mathbb{E}^*)$ is the set of collection of dual edges that form closed loops.
- Remark: we can recover this expansion simply by using the bijection $\iota : \sigma \to \{e \in \mathbb{E}^*, e \text{ is a disorder edge}\}$ seeing that $H(\sigma) = 2\#\iota(\sigma) - \#\mathbb{E}$, hence
$$Z = \sum_{\mathcal{E}^* \in \mathcal{C}(\mathbb{E}^*)} e^{-\beta H(\iota^{-1}(\mathcal{E}^*))} = e^{\beta \#\mathbb{E}} \sum_{\mathcal{E}^* \in \mathcal{C}(\mathbb{E}^*)} e^{-2\beta \#\mathcal{E}^*}.$$
- The correlations are more complicated to obtain, yet, note that σ_x can be recovered from $\iota(\sigma)$ by considering $(-1)^{N_x^{\iota(\sigma)}}$, where $N_x^{\iota(\sigma)} := \#\{\gamma \in \iota(\sigma),\ \gamma \text{ loop which surrounds } x\}$, hence:
$$\mathbb{E}_+[\sigma_x] = \frac{\sum_{\mathcal{E}^* \in \mathcal{C}(\mathbb{E}^*)} (-1)^{N_x^{\mathcal{E}^*}} e^{-\beta H(\iota^{-1}(\mathcal{E}^*))}}{\sum_{\mathcal{E}^* \in \mathcal{C}(\mathbb{E}^*)} e^{-\beta H(\iota^{-1}(\mathcal{E}^*))}} = \frac{\sum_{\mathcal{E}^* \in \mathcal{C}(\mathbb{E}^*)} (-1)^{N_x^{\mathcal{E}^*}} e^{-2\beta \#\mathcal{E}^*}}{\sum_{\mathcal{E}^* \in \mathcal{C}(\mathbb{E}^*)} e^{-2\beta \#\mathcal{E}^*}}.$$

1.4.3. *High-Temperature Expansion.*

- We consider the free boundary condition, the case of + b.c. will be provided at the end of the section.
- We use the fact that $f(\sigma) = \cosh(\beta)[1 + \tanh(\beta)\sigma]$: from the general strategy (Section 1.4.1), with $f_0(\sigma) = 1$ and $f_1(\sigma) = \sigma$, and using the bijection between $(i_e)_{e \in E} \in \{0,1\}^{\mathbb{E}}$ and subsets $\mathcal{E}$ of $\mathbb{E}$,
$$Z_A = \cosh(\beta)^{\#\mathbb{E}} \sum_{\mathcal{E} \subset \mathbb{E}} \tanh(\beta)^{\#\mathcal{E}} \sum_{\sigma \in \{+1,-1\}^{\mathbb{V}}} \prod_{(x,y)=e \in \mathcal{E}} \sigma_x \sigma_y \sigma_A.$$
- For any $\mathcal{E} \subset \mathbb{E}$, $\prod_{(x,y)=e\in\mathcal{E}} \sigma_x\sigma_y\sigma_A = \prod_{x \in \mathbb{V}} \sigma_x^{n_{\mathcal{E}}(x)} \sigma_A$ where $n_{\mathcal{E}}(x) = \#\{e \in \mathcal{E}, x \text{ is an endpoint of } e\}$ is the degree of x in $\mathcal{E}$. Using the fact that $\sigma_x^{n_{\mathcal{E}}(x)} = 1$ if $n_{\mathcal{E}}(x)$ is even, and $\sigma_x^{n_{\mathcal{E}}(x)} = \sigma$ is it is odd, $\prod_{(x,y)=e\in\mathcal{E}} \sigma_x\sigma_y\sigma_A = \sigma_{\partial\mathcal{E}\Delta A}$ where:
  - $\partial\mathcal{E}$ is the source of $\mathcal{E}$, i.e. the set of x such that $n_{\mathcal{E}}(x)$ is odd,
  - $\sigma_C = \prod_{x \in C} \sigma_x$ for any $C \subset \mathbb{V}$,
  - Δ is the symetric difference: $A\Delta B = (A \setminus B) \cap (B \setminus A)$.
- *Orthogonality* of the σ_C.
  - Let $C \subset \mathbb{V}$: $\sum_{x\in\{+1,-1\}^{\mathbb{V}}} \sigma_C = \delta_{C=\emptyset} 2^{\mathbb{V}}$. Indeed, if $C = \emptyset$, this a consequence of the fact $\#\{+1,-1\}^{\mathbb{V}} = 2^{\mathbb{V}}$. If $C \neq \emptyset$, consider $x_0 \in C$, and the map $\sigma \to \iota(\sigma)$ where $\iota(\sigma)$ differs from σ only at x_0 (this is an involution, thus a bijection). Then $\iota(\sigma)_C = -\sigma_C$ and thus $\sum_{x\in\{+1,-1\}^{\mathbb{V}}} \sigma_C = \frac{1}{2}\left[\sum_{x\in\{+1,-1\}^{\mathbb{V}}} \sigma_C + \sum_{x\in\{+1,-1\}^{\mathbb{V}}} \sigma_C\right] = \frac{1}{2}\left[\sum_{x\in\{+1,-1\}^{\mathbb{V}}} (\sigma_C + \iota(\sigma)_C)\right] = 0$.
  - More generally, if $A, B \subset \mathbb{V}$,
$$\sum_{x\in\{+1,-1\}^{\mathbb{V}}} \sigma_A\sigma_B = \delta_{A=B} 2^{\mathbb{V}}.$$
Indeed, $\sigma_A\sigma_B = \sigma_{A\Delta B}$ and we can apply the previous result.
- The High temperature expansion: since $Z_A = \cosh(\beta)^{\#\mathbb{E}} \sum_{\mathcal{E}\subset\mathbb{E}} \tanh(\beta)^{\#\mathcal{E}} \sum_{\sigma\in\{+1,-1\}^{\mathbb{V}}} \sigma_{\partial\mathcal{E}\Delta A}$, using the previous result:
$$Z_A = 2^{\#\mathbb{V}} \cosh(\beta)^{\#\mathbb{E}} \sum_{\mathcal{E}\subset\mathbb{E}, \partial\mathcal{E}=A} \tanh(\beta)^{\#\mathcal{E}}.$$
- In particular, we get the high temperature expansion of the correlations for the free boundary condition:
$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A] = \frac{\sum_{\mathcal{E}\subset\mathbb{E},\partial\mathcal{E}=A} \tanh(\beta)^{\#\mathcal{E}}}{\sum_{\mathcal{E}\subset\mathbb{E},\partial\mathcal{E}=\emptyset} \tanh(\beta)^{\#\mathcal{E}}}.$$
- Geometric interpretation: $\partial\mathcal{E} = A$ if and only if we can write $\mathcal{E}$ as the disjoint union of loops and n paths which joins pairwise the elements of A. In particular, we recover the fact that if $\#A$ is odd, $\mathbb{E}_f^{\mathbb{G}}[\sigma_A] = 0$.
- Using the same arguments, we get the following high temperature expansion of the correlations for the + boundary condition:
$$\mathbb{E}_+^{\mathbb{G}}[\sigma_A] = \frac{\sum_{\mathcal{E}\subset\mathbb{E},\partial\mathcal{E}\Delta A\subset\partial\mathbb{G}} \tanh(\beta)^{\#\mathcal{E}}}{\sum_{\mathcal{E}\subset\mathbb{E},\partial\mathcal{E}\subset\partial\mathbb{G}} \tanh(\beta)^{\#\mathcal{E}}}.$$
Geometric interpretation: $\partial\mathcal{E}\Delta A \subset \partial\mathbb{G}$ if and only if we can write $\mathcal{E}$ as the disjoint union of loops, arcs between points in $\partial\mathbb{G}$ and paths which join pairwise the elements of $A \cup \mathcal{X}$ where $\mathcal{X} \subset \partial\mathbb{G}$.

- Exercise: prove this.

1.4.4. *FK-Expansion.*

- See Exercise Sheet 12, Exercise 2.

1.4.5. *Current Expansion.*

- We consider the free boundary conditions, the case of + b.c. can be studied in a similar way.
- We use the fact that $f(\sigma) = \sum_{n\in\mathbb{N}} \frac{\beta^n}{n!}$: from the general strategy (Section 1.4.1), with $f_i(\sigma) = \sigma^i$, we get:

$$Z_A = \sum_{(\mathrm{n}_e)_{e\in\mathbb{E}}\in\mathbb{N}^{\mathbb{E}}} \left(\prod_{e\in E} \frac{\beta^{\mathrm{n}_e}}{\mathrm{n}_e!}\right) \sum_{\sigma\in\{+1,-1\}^{\mathbb{V}}} \prod_{(x,y)=e\in\mathcal{E}} (\sigma_x\sigma_y)^{\mathrm{n}_e}\,\sigma_A.$$

- Notations/Definitions:
  - $\mathrm{n} := (\mathrm{n}_e)_{e\in\mathbb{E}} \in \mathbb{N}^{\mathbb{E}}$ is called a current,
  - $\hat{\mathrm{n}}_e = 1_{\{\mathrm{n}_e>0\}}$, $(\hat{\mathrm{n}}_e)_{e\in\mathbb{E}}$ is the skeleton of n.
  - with $\mathrm{n}(x) = \sum_{x \text{ is an endpoint of } e} \mathrm{n}_e$ we define the source $\partial\mathrm{n}$ of n as the set of x such that $\mathrm{n}(x)$ is odd.
- For any current n, $\prod_{(x,y)=e\in\mathcal{E}} (\sigma_x\sigma_y)^{\mathrm{n}_e}\,\sigma_A = \prod_{x\in\mathbb{V}} \sigma_x^{\mathrm{n}(x)}\sigma_A = \sigma_{\partial\mathrm{n}\Delta A}$. Using the orthogonality of the σ_C's, we get

$$\sum_{\sigma\in\{+1,-1\}^{\mathbb{V}}} \prod_{(x,y)=e\in\mathcal{E}} (\sigma_x\sigma_y)^{\mathrm{n}_e}\,\sigma_A = 2^{\#\mathbb{V}}\delta_{\partial\mathrm{n}\Delta A=\emptyset}.$$

- Thus, we get the current expansion for the (modified) partition function for the Ising model with free boundary condition:

$$Z_A = 2^{\#\mathbb{V}} \sum_{\mathrm{n}\in\mathbb{N}^{\mathbb{E}}|\partial\mathrm{n}\Delta A=\emptyset} \omega(\mathrm{n})$$

where $\omega(\mathrm{n}) = \left(\prod_{e\in E} \frac{\beta^{\mathrm{n}_e}}{\mathrm{n}_e!}\right)$, and thus the current expansion of the correlations for the free boundary condition:

$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A] = \frac{\sum_{\mathrm{n}\in\mathbb{N}^{\mathbb{E}}|\partial\mathrm{n}\Delta A=\emptyset}\omega(\mathrm{n})}{\sum_{\mathrm{n}\in\mathbb{N}^{\mathbb{E}}|\partial\mathrm{n}=\emptyset}\omega(\mathrm{n})}.$$

1.4.6. *Double current representation.*

- Goal: express $\mathbb{E}_f^{\mathbb{G}}[\sigma_A]^2$ as a probability of an event in a model with two random currents.
- Notations: recall $\hat{\mathrm{n}}_e = 1_{\{\mathrm{n}_e>0\}}$, $(\hat{\mathrm{n}}_e)_{e\in\mathbb{E}}$ is the skeleton of n; if $A\subset\mathbb{V}$ and $E\subset\mathbb{E}$, we denote $E\in\mathfrak{F}_A$ if and only if each connected component of the graph obtained by considering the vertices $\mathbb{V}$ but only the edges E intersects A an even number of times (for example, if $A=\{x,y\}$ this is equivalent to the fact that x and y are connected using concatenations of edges in E).
- **Tool: the switching lemma:** for any A and B subsets of $\mathbb{V}$, for any function F on currents,

$$\sum_{\substack{\partial\mathrm{n}_1=A\\ \partial\mathrm{n}_2=B}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2)F(\mathrm{n}_1+\mathrm{n}_2) = \sum_{\substack{\partial\mathrm{n}_1=\emptyset\\ \partial\mathrm{n}_2=A\Delta B}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2)1_{\{\widehat{\mathrm{n}_1+\mathrm{n}_2}\in\mathfrak{F}_A\}}F(\mathrm{n}_1+\mathrm{n}_2)$$

- Proof when $A=\{x,y\}$:
  - $\omega(\mathrm{n}_1)\omega(\mathrm{n}_2) = \frac{\beta^{\mathrm{n}_1}}{\mathrm{n}_1!}\frac{\beta^{\mathrm{n}_2}}{\mathrm{n}_2!} = \frac{\beta^{\mathrm{n}_1+\mathrm{n}_2}}{(\mathrm{n}_1+\mathrm{n}_2)!}\frac{(\mathrm{n}_1+\mathrm{n}_2)!}{\mathrm{n}_1!\mathrm{n}_2!} = \omega(\mathrm{n}_1+\mathrm{n}_2)\begin{pmatrix}\mathrm{n}_1+\mathrm{n}_2\\ \mathrm{n}_1\end{pmatrix}$.
  - Express everything using n_1 and $\mathrm{m}=\mathrm{n}_1+\mathrm{n}_2$:
    * $\sum_{\substack{\partial\mathrm{n}_1=A\\ \partial\mathrm{n}_2=B}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2)F(\mathrm{n}_1+\mathrm{n}_2) = \sum_{\substack{\partial\mathrm{n}_1=A,\mathrm{n}_1\le\mathrm{m}\\ \partial\mathrm{m}=A\Delta B}} \omega(\mathrm{m})\begin{pmatrix}\mathrm{m}\\ \mathrm{n}_1\end{pmatrix}F(\mathrm{m})$,
    * $\sum_{\substack{\partial\mathrm{n}_1=\emptyset\\ \partial\mathrm{n}_2=A\Delta B}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2)1_{\{\widehat{\mathrm{n}_1+\mathrm{n}_2}\in\mathfrak{F}_A\}}F(\mathrm{n}_1+\mathrm{n}_2) = \sum_{\substack{\partial\mathrm{n}_1=\emptyset,\mathrm{n}_1\le\mathrm{m}\\ \partial\mathrm{m}=A\Delta B}} \omega(\mathrm{m})\begin{pmatrix}\mathrm{m}\\ \mathrm{n}_1\end{pmatrix}1_{\{\widehat{\mathrm{m}}\in\mathfrak{F}_A\}}F(\mathrm{m})$,
  - It remains to show that

$$\sum_{\substack{\partial\mathrm{n}_1=A,\\ \mathrm{n}_1\le\mathrm{m}}} \begin{pmatrix}\mathrm{m}\\ \mathrm{n}_1\end{pmatrix} = 1_{\{\widehat{\mathrm{m}}\in\mathfrak{F}_A\}} \sum_{\substack{\partial\mathrm{n}_1=\emptyset,\\ \mathrm{n}_1\le\mathrm{m}}} \begin{pmatrix}\mathrm{m}\\ \mathrm{n}_1\end{pmatrix}.$$

- The condition $\{\widehat{\mathrm{m}} \in \mathfrak{F}_A\}$ is equivalent to the fact that x and y are connected in $\widehat{\mathrm{m}}$. If it is not the case, then there can not be any $\mathrm{n}_1 \leq \mathrm{m}$ with $\partial \mathrm{n}_1 = \{x, y\}$ since the last condition implies that x and y are connected in $\widehat{\mathrm{n}_1}$ hence they should be connected in $\widehat{\mathrm{m}}$. We assume that x and y are connected in $\widehat{\mathrm{m}}$: we need to prove
$$\sum_{\substack{\partial \mathrm{n}_1 = A,\\ \mathrm{n}_1 \leq \mathrm{m}}} \begin{pmatrix} \mathrm{m} \\ \mathrm{n}_1 \end{pmatrix} = \sum_{\substack{\partial \mathrm{n}_1 = \emptyset,\\ \mathrm{n}_1 \leq \mathrm{m}}} \begin{pmatrix} \mathrm{m} \\ \mathrm{n}_1 \end{pmatrix}.$$
- We consider the modified graph $\mathbb{G}_{\mathrm{m}}$ where each edge e is drawn m_e times instead of 1 time. Then $\sum_{\substack{\partial \mathrm{n}_1 = A,\\ \mathrm{n}_1 \leq \mathrm{m}}} \begin{pmatrix} \mathrm{m} \\ \mathrm{n}_1 \end{pmatrix}$ is the cardinality of the subgraphs $\mathbb{H} \subset \mathbb{G}_{\mathrm{m}}$ with source A (to a subgraph one can naturally associate a current on $\mathbb{G}$ by counting the number of times one uses the edge e) and $\sum_{\substack{\partial \mathrm{n}_1 = \emptyset,\\ \mathrm{n}_1 \leq \mathrm{m}}} \begin{pmatrix} \mathrm{m} \\ \mathrm{n}_1 \end{pmatrix}$ is the cardinality of the subgraphs $\mathbb{H} \subset \mathbb{G}_{\mathrm{m}}$ with no source.
- Since x and y are connected in $\widehat{\mathrm{m}}$, we can consider a path p in $\mathbb{G}_{\mathrm{m}}$ which connects x and y. Then the map $\mathbb{H} \to \mathbb{H} \oplus p = \{e \in \mathbb{H}\} \Delta \{e \in p\}$ is a bijection (since it is an involution) between the set of subgraphs of $\mathbb{G}_{\mathrm{m}}$ without source and the ones with source $\{x, y\}$. This allows us to conclude.

- Theorem: We consider the measure $\mathbb{P}^{\emptyset}(\mathrm{n}) = \frac{\omega(\mathrm{n})}{\sum_{\mathrm{n}, \partial \mathrm{n} = \emptyset} \omega(\mathrm{n})}$ on the currents with no source and the measure $\mathbb{P}^{\emptyset} \otimes \mathbb{P}^{\emptyset}(\mathrm{n}_1, \mathrm{n}_2) = \mathbb{P}^{\emptyset}(\mathrm{n}_1)\mathbb{P}^{\emptyset}(\mathrm{n}_2)$ (i.e. two independant currents). The Double Current representation of the square of the correlations is:
$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A]^2 = \mathbb{P}^{\emptyset} \otimes \mathbb{P}^{\emptyset}\left[\widehat{\mathrm{n}_1 + \mathrm{n}_2} \in \mathfrak{F}_A\right].$$
- Proof:
  - We have
$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A]^2 = \frac{\sum_{\substack{\partial \mathrm{n}_1 = A\\ \partial \mathrm{n}_2 = A}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2)}{\sum_{\substack{\partial \mathrm{n}_1 = \emptyset\\ \partial \mathrm{n}_2 = \emptyset}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2)}.$$
  - Using the switching lemma
$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A]^2 = \frac{\sum_{\substack{\partial \mathrm{n}_1 = \emptyset\\ \partial \mathrm{n}_2 = \emptyset}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2) 1_{\{\widehat{\mathrm{n}_1 + \mathrm{n}_2} \in \mathfrak{F}_A\}}}{\sum_{\substack{\partial \mathrm{n}_1 = \emptyset\\ \partial \mathrm{n}_2 = \emptyset}} \omega(\mathrm{n}_1)\omega(\mathrm{n}_2)}.$$
  - We conclude using the definition of the current distribution.

1.5. **The Magnetisation.** We will use all the tools developped in the previous section in order to prove the assertions in Section 1.1.4.

1.5.1. *The G-K-S inequality.*

- For the free boundary condition Ising model,
$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A \sigma_B] \geq \mathbb{E}_f^{\mathbb{G}}[\sigma_A]\,\mathbb{E}_f^{\mathbb{G}}[\sigma_B]$$
where we recall that $\sigma_A = \prod_{x \in A} \sigma_x$.
- Remark: this is similar to the FKG inequality we have seen for the percolation, but notice that σ_A is not an increasing function, neither σ_B. Note that it can be written in the following form:
$$\mathrm{Cov}_f^{\mathbb{G}}(\sigma_A, \sigma_B) \geq 0.$$
- The same holds for the + boundary condition (Exercise: do it using Section 1.2.1).
- The proof is based on the current expansion and the switching lemma:
  - Current expansion: $\mathbb{E}_f^{\mathbb{G}}[\sigma_A] = \frac{\sum_{\mathrm{n} \partial \mathrm{n} = A} w(\mathrm{n})}{\sum_{\mathrm{n} \partial \mathrm{n} = \emptyset} w(\mathrm{n})}$, where $w(\mathrm{n}) = \prod_{e \in \mathbb{E}} \frac{\beta^{\mathrm{n}_e}}{\mathrm{n}_e!}$.
  - Expand product: $\mathbb{E}_f^{\mathbb{G}}[\sigma_A]\,\mathbb{E}_f^{\mathbb{G}}[\sigma_B] = \frac{\sum_{(\mathrm{n}_1, \mathrm{n}_2) \in \mathrm{C_A} \times \mathrm{C_B}} w(\mathrm{n}_1) w(\mathrm{n}_2)}{\left(\sum_{\mathrm{n} \partial \mathrm{n} = \emptyset} w(\mathrm{n})\right)^2}$, where $\mathrm{C_A}$ is the set of currents with source $\partial \mathrm{n} = A$.
  - Switching lemma:
$$\sum_{(\mathrm{n}_1, \mathrm{n}_2) \in \mathrm{C_A} \times \mathrm{C_B}} w(\mathrm{n}_1) w(\mathrm{n}_2) = \sum_{(\mathrm{n}_1, \mathrm{n}_2) \in \mathrm{C}_{\emptyset} \times \mathrm{C}_{\mathrm{A} \Delta \mathrm{B}}} w(\mathrm{n}_1) w(\mathrm{n}_2) 1_{\{\widehat{n_1 + \mathrm{n}_2} \in \mathfrak{F}_A\}}$$

hence

$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A]\,\mathbb{E}_f^{\mathbb{G}}[\sigma_B] = \frac{\sum_{(\mathrm{n}_1,\mathrm{n}_2)\in \mathrm{C}_\emptyset\times \mathrm{C}_{\mathrm{A}\Delta\mathrm{B}}} w(\mathrm{n}_1)w(\mathrm{n}_2)1_{\{\widehat{n_1+\mathrm{n}_2}\in\mathfrak{F}_A\}}}{\left(\sum_{\mathrm{n}\,\partial\mathrm{n}=\emptyset} w(\mathrm{n})\right)^2}.$$

– Forget about the condition $\{\widehat{n_1+\mathrm{n}_2}\in\mathfrak{F}_A\}$:

$$\mathbb{E}_f^{\mathbb{G}}[\sigma_A]\,\mathbb{E}_f^{\mathbb{G}}[\sigma_B] \le \frac{\sum_{(\mathrm{n}_1,\mathrm{n}_2)\in \mathrm{C}_\emptyset\times \mathrm{C}_{\mathrm{A}\Delta\mathrm{B}}} w(\mathrm{n}_1)w(\mathrm{n}_2)}{\left(\sum_{\mathrm{n}\,\partial\mathrm{n}=\emptyset} w(\mathrm{n})\right)^2} = \frac{\sum_{\mathrm{n}\in \mathrm{C}_{\mathrm{A}\Delta\mathrm{B}}} w(\mathrm{n})}{\left(\sum_{\mathrm{n}\,\partial\mathrm{n}=\emptyset} w(\mathrm{n})\right)} = \mathbb{E}_f^{\mathbb{G}}[\sigma_{A\Delta B}].$$

– $\mathbb{E}_f^{\mathbb{G}}[\sigma_{A\Delta B}] = \mathbb{E}_f^{\mathbb{G}}[\sigma_A\sigma_B]$, hence $\mathbb{E}_f^{\mathbb{G}}[\sigma_A]\,\mathbb{E}_f^{\mathbb{G}}[\sigma_B] \le \mathbb{E}_f^{\mathbb{G}}[\sigma_A\sigma_B]$.

1.5.2. *Monotonicity with respect to the boundary condition.*

- We can couple Ising models with arbitrary boundary conditions b, with + b.c. and with − b.c. such that $\sigma^{(-)}\le\sigma^{(b)}\le\sigma^{(+)}$ (where $\sigma\le\rho$ means $\sigma_x\le\rho_x$ for all $x\in\mathbb{V}$). The proof is as follows:
  – We couple the heat-bath dynamics $\sigma_n^{(+)}$, $\sigma_n^{(-)}$ with + and − boundary conditions (we do not flip the boundary spins), together with $\sigma_n^{(b)}$ such that at any time $n\ge 0$, $\sigma_n^{(-)}\le\sigma_n^{(b)}\le\sigma_n^{(+)}$:
    * Consider the same vertex at each step,
    * Decide the sign of the spin using the same uniform variable. ($\sigma_{n+1}^{(b/+/-)}(x) = 1$ if $U \le \frac{e^{-\beta H_+}}{e^{-\beta H_+}+e^{-\beta H_-}}$ where H_+ is the energy of the configuration obtained from $\sigma_n^{(b/+/-)}$ by putting the spin at x equal to $+1$, and similarly for H_- putting the spin at x equal to -1).
  – From Question 3 of Exercise 3 of Ex. Sheet 12, if two starting configurations are such that $\sigma_0\le\tilde\sigma_0$, then for all the times steps $\sigma_n\le\tilde\sigma_n$ for the Heat bath dynamics.
  – We initialize the Heat bath dynamics using:
    * for the − b.c., the constant configuration $\sigma_0^{(-)}(x)=-1$
    * for the + b.c., the constant configuration $\sigma_0^{(-)}(x)=+1$
    * for the b b.c., the configuration where $\sigma_x=b(x)$ on the boundary and $\sigma_x=-1$ for all inner vertices.
  – The starting configurations are such that $\sigma_0^{(-)}\le\sigma_0^{(b)}\le\sigma_0^{(+)}$, and thus, for any n,

$$\sigma_n^{(-)}\le\sigma_n^{(b)}\le\sigma_n^{(+)}.$$

  Taking the limit $n\to 0$, and using the convergence theorem, we get a coupling for the Ising measures for different boundary conditions:

$$\sigma^{(-)}\le\sigma^{(b)}\le\sigma^{(+)}.$$

- For any increasing function $F:\{+1,-1\}^{\mathbb{V}_{in}}\to\mathbb{R}$, (i.e. if $\sigma\le\rho$ then $F(\sigma)\le F(\rho)$) then

$$\mathbb{E}^-[F(\sigma)]\le\mathbb{E}^b[F(\sigma)]\le\mathbb{E}^+[F(\sigma)]$$

- We consider a square-grid discretization Ω_δ with mesh δ of a planar domain with $0\in\Omega$, and with boundary $\partial\Omega_\delta$. The function $\sigma\mapsto\sigma_0$ is an increasing function:

$$\mathbb{E}_\delta^-[\sigma_0]\le\mathbb{E}_\delta^b[\sigma_0]\le\mathbb{E}_\delta^+[\sigma_0].$$

1.5.3. *Monotonicity with respect to β (see also Ex 2, Sheet 11).*

- With the notations $Z=\sum_\sigma e^{-\beta H(\sigma)}$ and $Z_F=\sum_\sigma F(\sigma)e^{-\beta H(\sigma)}$, one has $\mathbb{E}_\beta[F(\sigma)]=\frac{Z_F}{Z}$ hence

$$\frac{d}{d\beta}\mathbb{E}_\beta[F(\sigma)] = \frac{\frac{d}{d\beta}Z_F}{Z} - \mathbb{E}[F(\sigma)]\frac{\frac{d}{d\beta}Z}{Z}$$

  Since $\frac{d}{d\beta}Z=-\beta\sum_\sigma H(\sigma)e^{-\beta H(\sigma)}=-\beta Z_H$ and $\frac{d}{d\beta}Z_F=-\beta\sum_\sigma F(\sigma)H(\sigma)e^{-\beta H(\sigma)}=-\beta Z_{FH}$, we get

$$\frac{d}{d\beta}\mathbb{E}_\beta[F(\sigma)] = -\beta\mathbb{E}_\beta[F(\sigma)H(\sigma)]+\beta\mathbb{E}_\beta[F(\sigma)]\,\mathbb{E}[H(\sigma)]$$

  or

$$\frac{d}{d\beta}\mathbb{E}_\beta[F(\sigma)] = -\beta[\mathrm{Cov}_\beta[F,H]]$$

- If we specify using the definition of the energy in the Ising model, we get
$$\frac{d}{d\beta}\mathbb{E}_{\beta}\left[F(\sigma)\right]=\beta\left[\sum_{e\in\mathbb{E}}\mathrm{Cov}_{\beta}\left[F,\sigma_e\right]\right]$$
where for any edge $e=\{x,y\}$, $\sigma_e:\sigma\to\sigma_x\sigma_y$.
- Theorem: For the + boundary condition, for any $A\subset\mathbb{V}_{\text{int}}$, $\beta\mapsto\mathbb{E}^{\mathbb{G}}_{+,\beta}\left[\sigma_A\right]$ is non decreasing.
  - By the previous results: $\frac{d}{d\beta}\mathbb{E}^{\mathbb{G}}_{+,\beta}\left[\sigma_A\right]=\beta\left[\sum_{e\in\mathbb{E}}\mathrm{Cov}^{\mathbb{G}}_{+,\beta}\left[\sigma_A,\sigma_e\right]\right]$
  - Using the G.K.S. inequality, $\frac{d}{d\beta}\mathbb{E}^{\mathbb{G}}_{+,\beta}\left[\sigma_A\right]\geq 0$.
- In particular, if we consider a square-grid discretization Ω_δ with mesh δ of a planar domain $0\in\Omega$, and with the natural boundary $\partial\Omega_\delta$, once we will have shown that $\langle\sigma_0\rangle^{\Omega}_{+,\beta}:=\lim_{\delta\to 0}\mathbb{E}^{\Omega_\delta}_{+,\beta}\left[\sigma_0\right]$ exists, this theorem shows that
$$\beta\mapsto\langle\sigma_0\rangle^{\Omega}_{+,\beta}$$
is non decreasing in β.
- Remark: the same holds for free boundary conditions.

1.5.4. *Monotonicity with respect to δ.*

- Theorem: If we consider the Ising model with + boundary condition on a square-grid discretization Ω_δ with mesh δ of a planar domain $0\in\Omega$, the function $\delta\mapsto\mathbb{E}^{\Omega_\delta}_{+}\left[\sigma_A\right]$ is increasing in δ (or in other words $\mathbb{E}^{\Omega_\delta}_{+}\left[\sigma_A\right]$ decreases as $\delta\searrow 0$: since the boundary is further away when $\delta\to 0$, the boundary effect gets smaller)
  - This is equivalent to the fact that if we consider $\Lambda'\subset\Lambda$ some induced graphs in $\mathbb{Z}^2$, then
$$\mathbb{E}^{\Lambda'}_{\beta,+}\left[\sigma_A\right]\geq\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_A\right].$$
  - The + b.c. Ising model on Λ' can be obtained from the one on Λ by conditionning the spins in $\Lambda\setminus\Lambda'$ to be equal to $+1$:
$$\mathbb{E}^{\Lambda'}_{\beta,+}\left[\sigma_A\right]=\frac{\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_A\prod_{x\in\Lambda\setminus\Lambda'}1_{\sigma_x=+1}\right]}{\mathbb{E}^{\Lambda}_{\beta,+}\left[\prod_{x\in\Lambda\setminus\Lambda'}1_{\sigma_x=+1}\right]}$$
  - We write this using the $\sigma_.$ functions: $1_{\sigma_x=1}=\frac{1}{2}\left(\sigma_x+1\right)$, thus $\prod_{x\in\Lambda\setminus\Lambda'}1_{\sigma_x=+1}=\frac{1}{2^{|\Lambda\setminus\Lambda'|}}\prod_{x\in\Lambda\setminus\Lambda'}\left(\sigma_x+1\right)$ and thus there exists (we do need to be explicit) $a_C\geq 0$ for any $C\subset\Lambda\setminus\Lambda$ such that:
$$\prod_{x\in\Lambda\setminus\Lambda'}1_{\sigma_x=+1}=\sum_{C\subset\Lambda\setminus\Lambda}a_C\sigma_C.$$
  - Insert this decomposition:
$$\mathbb{E}^{\Lambda'}_{\beta,+}\left[\sigma_A\right]=\frac{\sum_{C\subset\Lambda\setminus\Lambda}a_C\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_A\sigma_C\right]}{\sum_{C\subset\Lambda\setminus\Lambda}a_C\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_C\right]}$$
  - Use GKS inequality:
$$\mathbb{E}^{\Lambda'}_{\beta,+}\left[\sigma_A\right]\geq\frac{\sum_{C\subset\Lambda\setminus\Lambda}a_C\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_A\right]\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_C\right]}{\sum_{C\subset\Lambda\setminus\Lambda}a_C\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_C\right]}=\mathbb{E}^{\Lambda}_{\beta,+}\left[\sigma_A\right].$$
- Remark: for free boundary condition, $\mathbb{E}^{\Omega_\delta}_{f}\left[\sigma_A\right]$ is increasing as $\delta\searrow 0$. The proof is different:
  - we prove that if $\Lambda'\subset\Lambda$ then $\mathbb{E}^{\Lambda'}_{\beta,f}\left[\sigma_A\right]\leq\mathbb{E}^{\Lambda}_{\beta,f}\left[\sigma_A\right]$. As for the + boundary condition, the Ising model on Λ' is a modification of the Ising model on Λ.
  - What one can do is to consider an "edge-dependent" inverse temperature: $(\tilde{\beta}_e)_{e\in\mathbb{E}}$. If we consider an Ising model on Λ, with $\tilde{\beta}_e=\beta$ if e is an edge in Λ' and $\tilde{\beta}_e=0$ if e is not an edge in Λ', then what we see is an Ising model on Λ' which is independent from a collection of i.i.d. variables σ_x for $x\in\Lambda\setminus\Lambda'$. Thus, $\mathbb{E}^{\Lambda'}_{\beta,f}\left[\sigma_A\right]=\mathbb{E}^{\Lambda}_{\tilde{\beta},f}\left[\sigma_A\right]$.
  - In order to go from the model with probability measure $\mathbb{P}^{\Lambda}_{\tilde{\beta},f}$ to the Ising model on Λ, one has to increase the inverse temperature on each edge $e\notin\Lambda'$.
  - Using the same arguments as in the previous section, the expectation of σ_A increases as we do so. Thus $\mathbb{E}^{\Lambda'}_{\beta,f}\left[\sigma_A\right]=\mathbb{E}^{\Lambda}_{\tilde{\beta},f}\left[\sigma_A\right]\leq\mathbb{E}^{\Lambda}_{\beta,f}\left[\sigma_A\right]$.

- Corollary: $\langle\sigma_0\rangle^{\Omega}_{+,\beta} := \lim_{\delta\to 0}\mathbb{E}^{\Omega_\delta}_{+,\beta}[\sigma_0]$ exists. (it is bounded below and decreases, so it converges).

1.5.5. *Peierls argument.*

- Theorem: If $\beta > 0$ is large, then $\langle\sigma_0\rangle^{\Omega}_{+,\beta} > 0$.
- The idea is that since we consider $\beta \gg 1$, we are in the low temperature regime and thus, we expect few disorder loops: we will use the low-temperature expansion. In this picture, we can take β large enough so that there is a high probability that 0 is not surrounded by any disorder loop and thus that the spin at 0 is $+1$. Indeed, the probability that 0 is surrounded by a given loop of length k will be of order $e^{-2\beta k}$ (i.e. super small) and the number of such loops is only of order $k4^k$.
- Formal proof:
  - we need to bound uniformly in δ, $\mathbb{E}^{\Omega_\delta}_{+,\beta}[\sigma_0] \geq C > 0$.
  - using the low temperature expansion: if N_0 is the number of disorder loops surrounding 0,
$$\mathbb{E}^{\Omega_\delta}_{+,\beta}[\sigma_0] = \mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 \text{ is even}] - \mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 \text{ is odd}] = 2\mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 \text{ is even}] - 1$$
  - Since $\mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 \text{ is even}] \geq \mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 = 0]$, if we show that $\mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 = 0] > \frac{1}{2}$ uniformly in δ we will have shown the theorem. Equivalently, we show that $\mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 < 0] < \frac{1}{2}$
  - Now
$$\begin{aligned}\mathbb{P}^{\Omega_\delta}_{+,\beta}[N_0 < 0] &= \mathbb{P}^{\Omega_\delta}_{+,\beta}[\cup_{\gamma \text{ loop surrounds } 0}\{\gamma \text{ is a disorder loop}\}] \\ &\leq \sum_{\gamma \text{ loop surrounds } 0} \mathbb{P}^{\Omega_\delta}_{+,\beta}[\gamma \text{ is a disorder loop}] \\ &\leq \sum_k \sum_{\gamma \text{ surrounds } 0,\text{ length } k} \mathbb{P}^{\Omega_\delta}_{+,\beta}[\gamma \text{ is a disorder loop}]\end{aligned}$$
  - We can bound $\sum_{\gamma \text{ surrounds } 0,\text{ length } k}\mathbb{P}^{\Omega_\delta}_{+,\beta}[\gamma \text{ is a disorder loop}]$:
    * if γ surround 0 and is of length k, $\mathbb{P}^{\Omega_\delta}_{+,\beta}[\gamma \text{ is a disorder loop}] \leq e^{-2\beta k}$
      · If σ is a configuration such that γ is a disorder loop (i.e. if $x \sim y$ and x is inside γ and y is outside, then $\sigma(x) \neq \sigma(y)$), then the energy of σ splits into [1]-the energy of σ inside γ, i.e. the energy of $\sigma_1 := \sigma_{|Int(\gamma)}$ [2]-the energy across γ [3]-the energy of σ outside γ, , i.e. the energy of $\sigma_2 := \sigma_{|Out(\gamma)}$:
$$H(\sigma) = H(\sigma_1) + H(\sigma_2) + |\gamma|$$
      where $|\gamma|$ is the length of γ.
      · We consider the spin-flip operation inside γ, i.e using the same notations, $\iota(\sigma)$ is equal to $-\sigma_1$ inside γ and σ_2 outside γ. It is a bijection between configurations where γ is a disorder loop, and configuration where all the spins across γ agrees (we will say that γ is an order loop). If γ is a disorder loop of σ:
$$H(\iota(\sigma)) = H(\sigma_1) + H(\sigma_2) - |\gamma| = H(\sigma) - 2|\gamma|$$
      and thus, if γ is an order loop of σ
$$H(\iota(\sigma)) = H(\sigma) + 2|\gamma|$$
      · Hence
$$\begin{aligned}\mathbb{P}^{\Omega_\delta}_{+,\beta}[\gamma \text{ is a disorder loop}] &= \sum_{\sigma|\gamma \text{ is a disorder loop}} \frac{e^{-\beta H(\sigma)}}{Z_\beta} \\ &= \sum_{\sigma|\gamma \text{ is an order loop}} \frac{e^{-\beta H(\iota(\sigma))}}{Z_\beta} \\ &= e^{-2\beta|\gamma|} \sum_{\sigma|\gamma \text{ is an order loop}} \frac{e^{-\beta H(\sigma)}}{Z_\beta} \leq e^{-2\beta|\gamma|}\end{aligned}$$
      since $\sum_{\sigma|\gamma \text{ is an order loop}} e^{-\beta H(\sigma)} \leq Z_\beta = \sum_\sigma e^{-\beta H(\sigma)}$.
    * the number of loops γ of length k which surrounds 0 is smaller than $k4^k$ (uniformly in δ)
      · consider the last point on the $y = 0$ axis where the loop crosses 0: $\sim \frac{k}{2} \leq k$ choices.
      · then the number of loops which begins at this point and of length k is saller than 4^k.

    - Thus $\sum_{\gamma \text{ surrounds } 0, \text{ length } k} \mathbb{P}[\gamma \text{ is a disorder loop}] \leq k4^k e^{-2\beta k}$
  - Thus $\mathbb{P}[N_0 < 0] \leq \sum_{k \geq 0} k4^k e^{-2\beta k}$ which can be made as small as one wants by considering β large enough.

1.5.6. *Dual Peierls argument.*

- Theorem: If $\beta > 0$ is small, then $\langle \sigma_0 \rangle_{+,\beta}^{\Omega} = 0$.
- The idea is that since we consider $\beta \ll 1$, we are in the high temperature regime and thus, we expect the "influence" lines between the spin at 0 and the +1 spins on the boundary to be very lengtly and thus not to contribute a lot: we will use the high-temperature expansion. In this picture, in order to reach the boundary, an "influence" line need to do at least $\sim 1/\delta$ steps, so if the overall contribution is finite, this truncation will lead to a vanishing magnetisation.
- Formal proof:
  - We show that $\lim_{\delta \to 0} \mathbb{E}_{+,\beta}^{\Omega_\delta}[\sigma_0] = 0$
  - Using the the high temperature expansion,
$$\mathbb{E}_{+,\beta}^{\Omega_\delta}[\sigma_0] = \frac{\sum_{\mathcal{E} \subset E, \partial\mathcal{E}\Delta\{0\} \subset \partial\Omega_\delta} (\tanh\beta)^{\#\mathcal{E}}}{\sum_{\mathcal{E} \subset E, \partial\mathcal{E} \subset \partial\Omega_\delta} (\tanh\beta)^{\#\mathcal{E}}}$$
  - For any $\mathcal{E} \subset E$, $\partial\mathcal{E}\Delta\{0\} \subset \partial\Omega_\delta$, there exists a unique path $p : 0 \to \partial\Omega_\delta$ and an unique subset $\tilde{\mathcal{E}} \subset E$ with $\partial\tilde{\mathcal{E}} \subset \partial\Omega_\delta$ such that $\mathcal{E} = \tilde{\mathcal{E}} \cup p$. Using this notation, $(\tanh\beta)^{\#\mathcal{E}} = (\tanh\beta)^{\#p} (\tanh\beta)^{\#\tilde{\mathcal{E}}}$ and thus, using this new parametrization:
$$\mathbb{E}_{+,\beta}^{\Omega_\delta}[\sigma_0] = \sum_{p:0\to\partial\Omega_\delta} \tanh^{\#p}(\beta) \frac{\sum_{\tilde{\mathcal{E}} \subset E\setminus\pi : \partial\tilde{\mathcal{E}} \subset \partial\Omega_\delta\setminus\pi} \tanh^{\#\tilde{\mathcal{E}}}(\beta)}{\sum_{\mathcal{E} \subset E : \partial E \subset \partial\Omega_\delta} \tanh^{\#\mathcal{E}}(\beta)}$$
    which implies, since $\sum_{\tilde{\mathcal{E}} \subset E\setminus\pi : \partial\tilde{\mathcal{E}} \subset \partial\Omega_\delta\setminus\pi} \tanh^{\#\tilde{\mathcal{E}}}(\beta) \leq \sum_{\mathcal{E} \subset E : \partial E \subset \partial\Omega_\delta} \tanh^{\#\mathcal{E}}(\beta)$ that
$$\mathbb{E}_{+,\beta}^{\Omega_\delta}[\sigma_0] \leq \sum_{p:0\to\partial\Omega_\delta} \tanh^{\#p}(\beta) \leq \sum_k \tanh^k(\beta)\#\{p : 0 \to \partial\Omega_\delta,\ |p| = k\}.$$
  - Now, we have
    - for any $p : 0 \to \partial\Omega_\delta$, when δ is small enough, the length $|p| \geq \frac{C}{\delta}$ where C is a constant which depends only on the domain Ω.
    - For any k, $\#\{p : 0 \to \partial\Omega_\delta : |p| \leq k\} \leq 4^k$.
  - So, $\mathbb{E}_{+,\beta}^{\Omega_\delta}[\sigma_0] \leq \sum_{k \geq C/\delta} [4\tanh(\beta)]^k$ and thus, if we choose β small enough, this goes to 0 as $\delta \to 0$.

1.5.7. *Kramer-Wannier duality.*

- It remains to understand what could be $\beta_c = \sup\left\{\beta, \langle\sigma_0\rangle_\beta^+ = 0\right\}$.
- Note that the high temperature expansion of Z (with free boundary conditions)
$$Z \propto \sum_{\mathcal{E} \subset \mathbb{E}, \partial\mathcal{E} = \emptyset} \tanh(\beta)^{\#\mathcal{E}}$$
  and the low temperature expansion of Z of the Ising model on the dual graph (with free boundary conditions)
$$Z_A \propto \sum_{\mathcal{E} \subset \mathbb{E}, \partial\mathcal{E} = \emptyset} \left(e^{-2\beta}\right)^{\#\mathcal{E}}$$
  look the same.
- Indeed, we have exchanged
  - The graph $\Omega_\delta^* \leftrightarrow \Omega_\delta$ and its dual
  - The boundary conditions $+ \leftrightarrow$ free
  - The parameter $e^{-2\beta} \leftrightarrow \tanh\beta$. Let's call β^* the parameter such that $\tanh(\beta^*) = e^{-2\beta}$: if β increases, β^* decreases.
- One can show that the critical point is actually the self-dual point $\beta = \beta^*$, i.e. when $\beta = \frac{1}{2}\ln\left(\sqrt{2}+1\right)$.

LATTICE MODELS (MA1, EPFL)

1. Dimer Model

1.1. Generalities about the dimer model.

- We consider random dimer tiling (i.e. domino covering or perfect matching) of a given graph.
- This model exhibits an extremely rich behavior and many more 'physical' models can be mapped to it.
- The dimer model is very well understood on *bipartite (i.e. 2-colorable) planar graphs.*
- This lecture is focused on the dimer model on square grid domains (subgraphs of $\mathbb{Z}^2$).

1.2. Number of domino tilings of a checkerboard: statement. Today, we want to prove a classical theorem about dimer counting [Kas61, FiTe61]:

- The number of domino tilings of an $p \times q$ checkerboard is given by (unless pq is odd)
$$\sqrt{\left|\prod_{j=1}^{p}\prod_{k=1}^{q} 2\cos\left(\frac{\pi j}{p+1}\right) + 2i\cos\left(\frac{\pi k}{q+1}\right)\right|}.$$
- The proof of this result will teach us interesting things about the dimer model.

1.3. Key steps of the proof: Two main parts:

- Write the number of domino tilings in terms of the determinant of a matrix.
- Diagonalize the matrix and take the product of the eigenvalues.

1.4. Preliminary result: number of domino tilings as a permanent.

- Consider a square grid domain G with black vertices $\mathcal{B} = \{b_1, \ldots, b_n\}$ and white vertices $\mathcal{W} = \{w_1, \ldots, w_n\}$ (we color the vertices in such a way that adjacent vertices are of different colors).
- (Reduced) adjacency matrix A of G: an $n \times n$ matrix $(a_{bw})_{b\in\mathcal{B}, w\in\mathcal{W}}$ indexed by the black/white vertices of G such that $a_{bw} = 1$ if $b \sim w$ and $a_{bw} = 0$ otherwise.
- Permanent of A: $\mathrm{Per}\,(A) = \sum_{\sigma\in\mathcal{S}_n} a_{b_1 w_{\sigma(1)}} \cdots a_{b_n w_{\sigma(n)}}$ (like the determinant without the signature).
- In the sum $\sum_{\sigma\in\mathcal{S}_n} a_{b_1 w_{\sigma(1)}} \cdots a_{b_n w_{\sigma(n)}}$, we get a nonzero term whenever $b_1 \sim w_{\sigma(1)}, \ldots, b_n \sim w_{\sigma(n)}$, i.e. $\langle b_1 w_{\sigma(1)}\rangle$, ..., $\langle b_n w_{\sigma(n)}\rangle$ is a dimer cover.
- So: $\mathrm{Per}\,(A) = \#$dimer covers (this works for any bipartite graph),
- Inconvenient: this not very useful. Unlike the determinant, the permanent is very hard to compute and does not have good properties.

1.5. Next step: number of domino tilings as a determinant.

- Kasteleyn matrix $K = (k_{bw})_{b\in\mathcal{B}, w\in\mathcal{W}}$ for the square grid (can be generalized to other planar graphs):
  - $k_{bw} = 1$ if $\langle bw\rangle$ is a horizontal edge,
  - $k_{bw} = i = \sqrt{-1}$ if $\langle bw\rangle$ is a vertical edge and
  - $k_{bw} = 0$ otherwise.
- Theorem: $|\det K| = \#$dimer covers of G.
  - Proof: we expand:
$$\det K = \sum_{\sigma\in\mathcal{S}_n} \epsilon(\sigma)\, k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}}.$$
  - As before, all nonzero terms correspond to a dimer tiling. To get the result, enough to show the following lemma:
- *Lemma:* Let σ and $\tilde{\sigma}$ be two permutation corresponding to nonzero terms. Then
$$\epsilon(\sigma)\, k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}} = \epsilon(\tilde{\sigma})\, k_{b_1 w_{\tilde{\sigma}(1)}} \cdots k_{b_n w_{\tilde{\sigma}(n)}}.$$
  - Let $\mathcal{T}$ and $\tilde{\mathcal{T}}$ be the corresponding dimer tilers. If we superpose $\mathcal{T}$ and $\tilde{\mathcal{T}}$ (i.e. take the XOR $\mathcal{T} \oplus \tilde{\mathcal{T}}$). we get a collection of loops on G.
  - We can move from $\mathcal{T}$ to $\tilde{\mathcal{T}}$ by 'rotating' the dimers of $\mathcal{T}$ along each loop.

- We can suppose that $\mathcal{T} \oplus \tilde{\mathcal{T}}$ just consists of one loop $b_{i_1} w_{i_1} \cdots b_{i_k} w_{i_k}$ and that
$$\langle b_{i_1} w_{i_1} \rangle, \ldots, \langle b_{i_k} w_{i_k} \rangle \in \mathcal{T} \quad \text{and} \quad \langle b_{i_1} w_{i_k} \rangle, \ldots, \langle b_{i_k} w_{i_1} \rangle \in \tilde{\mathcal{T}}.$$
- We have that $\sigma = (i_1 \ \ldots \ i_k) \circ \tilde{\sigma}$ and $\epsilon(\sigma) = (-1)^{k+1} \epsilon(\tilde{\sigma})$.
- To get the desired result
$$\epsilon(\sigma) k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}} = \epsilon(\tilde{\sigma}) k_{b_1 w_{\tilde{\sigma}(1)}} \cdots k_{b_n w_{\tilde{\sigma}(n)}}$$
one should check that
$$k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}} = (-1)^{k+1} k_{b_1 w_{\tilde{\sigma}(1)}} \cdots k_{b_n w_{\tilde{\sigma}(n)}}.$$
- Provided by the following lemma, using that the number of vertices inside the loop $b_{i_1} w_{i_1} \cdots b_{i_k} w_{i_k}$ is even (it can be tiled by dimers):

- *Lemma:* for any cycle $b_1 w_1 \ldots b_k w_k$, if $m_1 = k_{b_1 w_1} k_{b_2 w_2} \ldots k_{b_k w_k}$ and $m_2 = k_{b_2 w_1} k_{b_3 w_2} \ldots k_{b_1 w_k}$, we have that $m_1 = (-1)^{\ell+k+1} m_2$, where ℓ is the number of vertices strictly inside the cycle.
  - Proof: by induction (check that when one adds a face to the domain inside the cycle, the property is maintained).

1.6. **Computing the determinant.**

- Now that we have $|\det K| = \#$dimer tilings, how to get the formula for the number of tilings of an $p \times q$ checkerboard, with $2n$ vertices?
- Consider the (extended) Kasteleyn matrix: it is the $2n \times 2n$ matrix $\tilde{K} = \begin{pmatrix} 0 & K \\ K^T & 0 \end{pmatrix}$.
  - $\tilde{K}$ is simply the matrix indexed by $\{b_1, \ldots, b_n, w_1, \ldots, w_n\}$ (in that order) such that $k_{vw} = 1$ if $\langle vw \rangle$ is a horizontal edge, $k_{vw} = i$ is $\langle vw \rangle$ is vertical and $k_{vw} = 0$ otherwise.
  - We have that $\left|\det \tilde{K}\right| = |\det K|^2$ (easy to see from the expansion of the determinant).
- It remains to show that:
$$\det \tilde{K} = \prod_{j=1}^{p} \prod_{k=1}^{q} \left(2 \cos\left(\frac{\pi j}{p+1}\right) + 2i \cos\left(\frac{\pi k}{q+1}\right)\right)$$
- How to show that?
  - Let us find $p \times q$ independent eigenvectors of $\tilde{K}$ and compute their eigenvalues.
  - Identify the vectors indexed by the vertices with *functions* defined on the checkerboard $\{1, \ldots, p\} \times \{1, \ldots, q\}$. We have
$$\left(\tilde{K} f\right)(x, y) = f(x+1, y) + f(x-1, y) + i\left(f(x, y+1) + f(x, y-1)\right),$$
where f is set to 0 when taken outside of the range.
  - For $j \in \{1, \ldots, p\}$ and $k \in \{1, \ldots, q\}$, let $z := e^{\frac{i\pi j}{p+1}}$ and $w := e^{\frac{i\pi k}{q+1}}$ and consider the vector
$$f^{jk}(x, y) := \left(z^x - z^{-x}\right)\left(w^y - w^{-y}\right) = -4 \sin\left(\frac{\pi j x}{p+1}\right) \sin\left(\frac{\pi k y}{q+1}\right),$$
notice that f^{jk} is zero when $x = 0, p+1$ or $y = 0, q+1$.
  - We have that $f^{jk}_{(x,y)}$ is an eigenvector of $\tilde{K}$, of eigenvalue $\lambda = z + \frac{1}{z} + i\left(w + \frac{1}{w}\right)$: one can see that
$$\left(\tilde{K} f\right)(x, y) = \lambda f^{jk}(x, y)$$
(we set $f^{jk}(x, y) = 0$ if either x or y go outside of the checker board)
  - Remark: we could have use the results from the Heat Equation Section:
    * We consider $M_n f(x) = f(x+1) + f(x-1)$ defined on the set of function on $\{0, \ldots, n+1\}$ with boundary conditions $f(0) = f(n+1) = 0$.
    * If f is an eigenvector of M_p, $M_p f = \lambda f$ and g is an eigenvector of M_q, $M_q g = \mu g$ then $\varphi(x, y) = f(x) g(y)$ is an eigenvector of $\tilde{K}$ with eigenvalue $\lambda + i\mu$.
    * We have already seen in the Heat Equation section that the eigenvalues of M_n are $2 \cos\left(\frac{\pi j}{n+1}\right)$, thus the eigenvalues of $\tilde{K}$ are $2 \cos\left(\frac{\pi j}{p+1}\right) + 2i \cos\left(\frac{\pi k}{q+1}\right)$.
  - Hence $\det \tilde{K}$ is the product of the eigenvalues (the eigenvectors are independent, as they have distinct eigenvalues).